11.1 $\int_{-r}^{0} f = \int_{-r}^{0} -f(-x)\,dx \overset{g(x)=-x}{=} \int_{-r}^{0} f(g(x))g'(x) \overset{\text{Substitution §5.8.1}}{=} \int_{g(-r)}^{g(0)} f = \int_{r}^{0} f = -\int_{0}^{r} f.$

$$\int_{-r}^{r} f = \int_{-r}^{0} f + \int_{0}^{r} f = -\int_{0}^{r} f + \int_{0}^{r} f = 0.$$

11.2

$\alpha \neq -1$: $\frac{x^{\alpha+1}}{\alpha+1}$ Stammfunktion von x^{α}

Beweis $f(x) = \frac{x^{\alpha+1}}{\alpha+1} = \frac{e^{\ln x(\alpha+1)}}{\alpha+1}$

$f'(x) = \frac{1}{\alpha+1}\left(e^{\ln x(\alpha+1)} \cdot (\alpha+1) \cdot \frac{1}{x}\right)$

$= x^{\alpha+1-1} = x^{\alpha}$

$$\int_{1}^{t} x^{\alpha} = \frac{1}{\alpha+1}\left(t^{\alpha+1} - 1\right)$$

$$\lim_{t\to\infty} \int_{1}^{t} x^{\alpha} = \lim_{t\to\infty} \frac{t^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} = \begin{cases} +\infty & \text{falls } \alpha+1>0 \\ -\frac{1}{\alpha+1} & \text{falls } \alpha+1<0 \end{cases}$$

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} x^{\alpha} = \lim_{\varepsilon\to 0}\left(\frac{1}{\alpha+1} - \frac{\varepsilon^{\alpha+1}}{\alpha+1}\right) = \begin{cases} \frac{1}{\alpha+1} & \text{falls } \alpha+1>0 \\ +\infty & \text{falls } \alpha+1<0 \end{cases}$$

$\alpha = 1$. $\log(t)$ Stammfunktion von x^{-1}

$$\lim_{t\to\infty} \int_{1}^{t} x^{-1} = \lim \log t = +\infty$$

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} x^{-1} = -\lim \log \varepsilon = +\infty$$

11.4 $\tan'(x) = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = 1 + \tan^2 x$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}.$$

$f(x) = x$

$$\int_{0}^{t} \underbrace{1}_{f'(x)} \cdot \arctan(x) = \left[x \arctan x\right]_{0}^{t} - \int_{0}^{t} \frac{x}{1+x^2}\,dx = \left[x\arctan x - \frac{1}{2}\log(1+x^2)\right]_{0}^{t}$$

Es folgt das $G(x) = x \arctan x - \frac{1}{2}\log(1+x^2)$ ist Stammfunktion von $\arctan(x)$.

11.4 $f(x) = li(x)$

$$g(x) = \frac{x}{\log(x)}$$

$$\lim_{x\to\infty} |g(x)| = \infty$$

$g'(x) = \dfrac{\log x - 1}{\log^2 x}$ hat keine Nullstelle in einer Umgebung $+\infty$

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim \frac{\frac{1}{\log x}}{\frac{\log x - 1}{\log^2 x}} = \lim_{x\to\infty} \frac{\log x}{\log x - 1} = 1.$$

Wir verwenden die Regel von de l'Hopital (§ 5.6.1).

$$\lim_{x\to\infty} \frac{\log(x)\, li(x)}{x} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 1$$

11.5 $\alpha \neq 0, \quad m \geq 2$

$$I_m(\alpha) = \int_{-1}^{1} (1-x^2)^m \cos(\alpha x)\, dx$$

z.z. $\alpha^2 I_m(\alpha) = 2m(2m-1) I_{m-1}(\alpha) - 4m(m-1) I_{m-2}.$

$f(x) = (1-x^2)^m. \quad g'(x) = \cos(\alpha x)$

$g(x) = \frac{\sin(\alpha x)}{\alpha}.$

$$I_m = \underbrace{(1-x^2)^m \frac{\sin \alpha x}{\alpha}\Big|_{-1}^{1}}_{=0} - \int_{-1}^{1} \frac{\sin \alpha x}{\alpha} \cdot m(1-x^2)^{m-1}(-2x)\,dx$$

$$= +\frac{2m}{\alpha} \int_{-1}^{1} (1-x^2)^{m-1} \sin(\alpha x)\, x$$

$$= \frac{2m}{\alpha} \left[\underbrace{\left[(1-x^2)^{m-1} x \left(\frac{-\cos \alpha x}{\alpha}\right)\right]_{-1}^{1}}_{=0} - \int_{-1}^{1} \frac{-\cos \alpha x}{\alpha} \cdot \left((m-1)(1-x^2)^{m-2}(-2x)x + (1-x^2)^{m-1}\right) \right]$$

$$= \frac{2m}{\alpha} \cdot \frac{1}{\alpha} \left[\int_{-1}^{1} \cos \alpha x\, (1-x^2)^{m-1} + \int (m-1)(-2x^2)(1-x^2)^{m-2} \cos \alpha x \right]$$

$$= \frac{2m}{\alpha^2} \left(I_{m-1} + (m-1) \int_{-1}^{1} (2-2x^2-2)(1-x^2)^{m-2} \cos \alpha x \right.$$

$$= \frac{2m}{\alpha^2} I_{m-1} + \frac{2m(m-1)}{\alpha^2} 2 I_{m-1} - \frac{2m(m-1)}{\alpha^2} 2 I_{m-2}$$

$$= \frac{2m(1+2m-2)}{\alpha^2} I_{m-1} - \frac{4m(m-1)}{\alpha^2} I_{m-2}.$$