Lie Groups
SoSe 2020 — Ubungsblatt 4

Ausgabe 02.06.20, **Abgabe** 16.06.19

Solutions are due on Tuesday 16nd June at 23:59. Please send it by email at

leonardo.patimo@math.uni-freiburg.de

Lie algebras and their representations

Aufgabe 4.1: Let G be a Lie group and let $g \in G$. The centralizer of g is the Lie subgroup

$$Z(g) := \{h \in G \mid hg = gh\}.$$

Show that the Lie algebra of $Z(g)$ is $\mathfrak{z}(g) := \{X \in \operatorname{Lie} G \mid Ad(g)X = X\}$.

(4 Punkte)

Aufgabe 4.2: Let G be a Lie group and let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ finite dimensional real representations of G.

- Show that $g \cdot (v_1 \otimes v_2) = (\rho_1(g)v_1) \otimes (\rho_2(g)v_2)$ defines a representation of G on $V_1 \otimes V_2$, which we denote by $\rho_1 \otimes \rho_2$.
- Show that $d(\rho_1 \otimes \rho_2)$ is the representation of $\operatorname{Lie} G$ on $V_1 \otimes V_2$ defined by

$$X \cdot (v_1 \otimes v_2) = (d\rho_1(X)v_1 \otimes v_2) + (v_1 \otimes d\rho_2(X)v_2).$$

(4 Punkte)

Aufgabe 4.3: Let V be a complex finite dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We denote by V_λ the eigenspaces of $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that, if V_0 and $V_1 = 0$, then $V = 0$.

Definition 1. A Lie algebra $\mathfrak{g}$ is called *simple* if it is not abelian and does contain any non-trivial ideal.

Bonus-Aufgabe 4.4: Let $\times$ be the cross product on $\mathbb{R}^3$ (recall that if $v, w \in \mathbb{R}^3$ and θ is the angle between v and w, then $v \times w$ is the unique vector in $\mathbb{R}^3$ of norm $\|v \times w\| = \|v\|\|w\| \sin(\theta)$, that is orthogonal to both v and w and such that $\det\big(v|w|v \times w\big) > 0$.

1. Show that $(\mathbb{R}^3, \times)$ is a Lie algebra isomorphic to $\mathfrak{so}_3(\mathbb{R})$.
2. Show that the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ is simple and that it does not contain any Lie subalgebra of dimension 2.
3. Deduce that the group $SU_2(\mathbb{C})$ does not have any irreducible real representation of dimension 2.

Hint: Otherwise we would get an isomorphism $\mathfrak{so}_3(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R})$ (why?), but these two Lie algebras are not isomorphic by (2)