LAST TIME Classification of irreducible complex rep. of S^1

$$\mathbb{Z} \longleftrightarrow \left\{ \begin{array}{l} \text{irr. complex} \\ \text{rep. of } S^1 \end{array} \right\}$$

$$n \longmapsto \left(\rho_n(e^{i\theta}) = e^{in\theta} \right)$$

Def Let V, W be reps. of a group G

then $V \oplus W$ is also a rep. of G

where $G \times (V \oplus W) \to V \oplus W$

$$(g, (v,w)) \longmapsto (g \cdot v, g \cdot w)$$

Def A completely reducible rep. of G is a rep. V

s.t. $V \cong \bigoplus_{i=1}^{n} V_i$, V_i irreducible

MASCHKE THEOREM For a compact Lie group G then all fin. dim. rep. are completely reducible

Obs We need G cpt.

$$\rho : (\mathbb{R}, +) \longrightarrow GL_2(\mathbb{R})$$

$$x \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ is not completely reducible}$$

S^1 cpt $\Rightarrow$ Maschke thm. hold!

EXAMPLE $L^2(S^1) = \{ f : S^1 \to \mathbb{C} \mid \int_{S^1} |f(e^{i\theta})|^2 d\theta < \infty \}$

$$S^1 \times L^2(S^1) \longrightarrow L^2(S^1)$$

$$(e^{i\theta}, f) \longmapsto (e^{i\theta}\cdot f)(e^{i\varphi}) = f(e^{i(\theta+\varphi)})$$

$V \subset L^2(S^1)$ fin. dim., stable under S^1

$\Rightarrow V$ is completely reducible

When

$$(e^{i\theta}\cdot f) = e^{in\theta} f \ ?$$

$$\forall \varphi \in \mathbb{R} \quad f(e^{i(\theta+\varphi)}) = e^{in\theta} f(e^{i\varphi})$$

set $\varphi = 0 \Rightarrow f(e^{i\theta}) = e^{in\theta} f(1)$

$$V_n = \langle e^{in\theta} \rangle, \quad V = \bigoplus_{\substack{n \in I \\ \subset \mathbb{Z}}} V_n$$

finite set.

Every $f \in L^2(S^1)$ can be approximated by an element $\bigoplus_{n\in\mathbb{Z}} V_n$

$$\Rightarrow f = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

$\Rightarrow$ FOURIER ANALYSIS

MATRIX GROUP (§1.2 FROM THE SCRIPT)

Let V be a fin. dim. real v. space

$GL(V) \cong GL_n(\mathbb{R})$ is an open subspace in $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$

Def A matrix group is a closed subgroup of $GL(V)$

Thm 1 Every matrix group is a smooth submanifold. In particular, it is a Lie group.

EXAMPLE

$$SL_n(\mathbb{R}) = \{A \in M_{n\times n} \mid \det A = 1\}$$

$$O_n(\mathbb{R}) = \{A \in M_{n\times n} \mid AA^t = Id\}$$
$$\parallel$$
$$\{\langle Av, Aw\rangle = \langle v, w\rangle \ \forall v, w\}$$

std. scalar product

$GL_n(\mathbb{C})$ is a closed subgroup of $GL_{2n}(\mathbb{R})$

The embedding takes $A + iB \in GL_n(\mathbb{C})$ to $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in GL_{2n}(\mathbb{R})$

So also $U_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \mid A\bar{A}^t = id\}$

A CRASH COURSE ON (EMBEDDED) DIFFERENTIABLE MANIFOLDS

Let X be an affine real space of fin. dim.

$X \cong \mathbb{R}^n$, $\vec{X}$ = space of vectors of X

$\vec{X} \cong \mathbb{R}^n$ as a vector space

$\vec{X} \times X \to X$

$(v,w) \mapsto v+w$

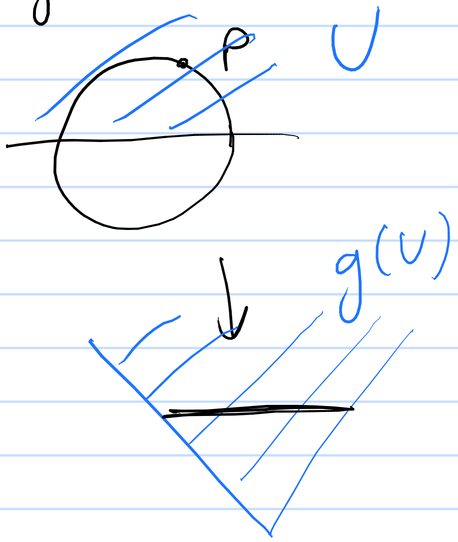
Def A subspace $M \subset X$ is a smooth manifold of dim k if $\forall p \in M$ $\exists (U,g)$ where U open nbhd of $p \in X$, $U \subset X$

and $g : U \xrightarrow{\sim} g(U)$ diffeomorphism, $g(U) \subset \mathbb{R}^n$ ← open set.

s.t. $g(U \cap M) = \{ z \in g(U) \mid z_{k+1} = \dots = z_n = 0 \}$

Here diffeo. $g \in C^\infty$, and $g^{-1} \in C^\infty$

EXAMPLE $S^1 \subset \mathbb{R}^2$



$g : U \to \mathbb{R}^2$

$(x,y) \mapsto (x, \sqrt{x^2+y^2} - 1)$

Prop $M \subset X$ is a dim k smooth mfld if and only if $\forall p \in M$ $\exists \varphi : W \longrightarrow X$, $\varphi \in \mathcal{C}^\infty$

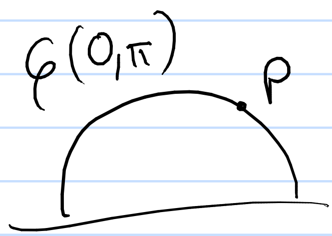
W open $\subset \mathbb{R}^k$

s.t.

1) $p \in \varphi(W)$ and $\varphi(W) \subset M$
2) $d_x\varphi$ is injective $\forall x \in W$
3) φ is a homeomorphism w/ its image

Def A pair (W, φ) as in the Prop a local chart of M at p.

EXAMPLE



$$\mathbb{R} \supset (0,\pi)$$

$$\varphi : (0,\pi) \longrightarrow \mathbb{R}^2$$
$$\theta \longmapsto e^{i\theta}$$

Def A set of charts (W_i, φ_i) of M such that

$M = \bigcup \varphi_i(W_i)$ is called atlas

Def M, N smooth mflds. with atlas (W_i, φ_i) and (V_j, ψ_j). Then a smooth morphism $f: M \to N$ is a continuous map s.t. $\forall i,j$

$\psi_j^{-1} \circ f \circ \varphi_i \in \mathcal{C}^\infty$ on $W_i \cap \varphi_i^{-1} \circ f^{-1} \circ \psi_j(V_j)$

The def. does not depend on the atlas that we choose.

Def A (embedded) Lie group G is a smooth mfld equipped w/ the structure of a group s.t.

$$\begin{array}{l} mlt: G \times G \to G \\ inv: G \to G \end{array} \quad \text{are smooth morphisms.}$$

EXAMPLES. S^1

- $GL_n(\mathbb{R})$ is an open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ so is a mfld in a trivial way (it has a unique global chart)

TANGENT SPACE

Def M^{C^∞} is a smooth mfld, $p \in M$

The tangent space at p is

$T_pM = \mathrm{Im}(d_u \varphi)_{C^\infty}$ where (W, φ) local chart at p, $u \in \varphi^{-1}(p)$

Rmk Does not depend on the chart.

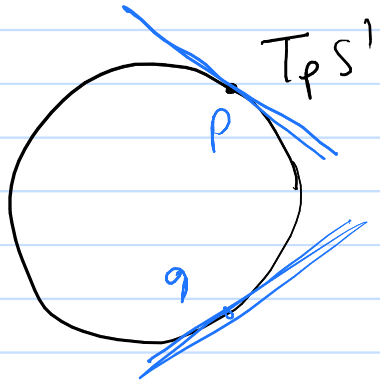
Equivalent definition

Take a $\mathcal{C}^\infty$ curve $\gamma : (-\varepsilon, \varepsilon) \to X \simeq \mathbb{R}^n$

$$\dot\gamma(0) = \left.\frac{d\gamma}{dt}\right|_{t=0} \in \vec{X}$$

$$T_pM = \mathrm{span} \left\langle \dot\gamma(0) \;\middle|\; \begin{array}{l} \gamma : (-\varepsilon, \varepsilon) \to X \text{ s.t.} \\ \mathrm{Im}\,\gamma \subset M, \text{ and } \gamma(0) = p \end{array} \right\rangle$$

EXAMPLE



- $T_p Gl_n(\mathbb{R}) \simeq M_{n\times n}(\mathbb{R})$.

BACK TO MATRIX GROUPS

Recall $A \in M_{n\times n}(\mathbb{R})$

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} \rightsquigarrow \text{easy to see that it converges}$$

If A, B commute, $\Rightarrow e^A \cdot e^B = e^{A+B}$.
$AB = BA$

If $B = -A$, $e^A \cdot e^{-A} = e^0 = Id \Rightarrow e^A \in Gl_n(\mathbb{R})$

Thm 2 Let $G \subset GL_n(\mathbb{R})$ be a matrix group. Then

1) $T_{Id} G = \{ A \in M_{n\times n}(\mathbb{R}) \mid \forall t \in \mathbb{R}, \exp(tX) \in G \}$

2) $\exists$ U open nbhd of $0 \in M_{n\times n}(\mathbb{R})$ s.t. $(U, \exp)$ is a local chart of G at Id.

2 for GL_n $\quad \exp : M_{n\times n}(\mathbb{R}) \to GL_n(\mathbb{R})$

$$A \mapsto e^A$$

is a local chart at 1

$d_0 \exp = Id \rightsquigarrow$ local chart at 1.

Thm 2 $\Rightarrow$ Thm 1 $\quad G \subset GL_n(\mathbb{R})$ matrix group

We have a local chart of G at Id

We can find a chart at g

$$g \cdot \exp : U \to G$$
$$A \mapsto g \cdot e^A$$

$\Rightarrow G$ is a smooth mfld

mult, inv are smooth morphisms. Easy to check!

$$\begin{array}{ccc} \text{inv}: G & \to & G \\ \downarrow & & \downarrow \\ GL_n(\mathbb{R}) & \to & GL_n(\mathbb{R}) \end{array}$$

Pf of thm 2

$$\mathfrak{g} = \left\{ A \in M_{n\times n}(\mathbb{R}) \;\middle|\; \begin{array}{l} \forall t \in \mathbb{R} \\ \exp(At) \in G \end{array} \right\}$$

CLAIM $\mathfrak{g}$ is a vector space.

- closed under scalar multiplication. ✓
- $A, B \in \mathfrak{g} \Rightarrow A+B \in \mathfrak{g}$.

TROTTER FORMULA

$$e^{A+B} = \lim_{n\to\infty} \Big(\underset{\substack{\cap \\ G}}{e^{\frac{A}{n}}} \cdot e^{\frac{B}{n}} \Big)^n$$

Sketch $e^{At} e^{Bt} = e^{(A+B)t + O(t^2)} \Longrightarrow$

$$\left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n = \left(e^{\frac{A+B}{n} + O\left(\frac{1}{n^2}\right)}\right)^n = e^{A+B + nO\left(\frac{1}{n^2}\right)} \xrightarrow{n\to\infty} e^{A+B}$$

$\mathfrak{g}$ sub vector space of $M_{n\times n}(\mathbb{R})$

Find a complement $\mathfrak{t}$

$$\mathfrak{g} \oplus \mathfrak{t} \cong M_{n\times n}(\mathbb{R})$$

$$\psi : \underset{\substack{\parallel \\ \mathfrak{g} \oplus \mathfrak{t}}}{M_{n\times n}(\mathbb{R})} \longrightarrow GL_n(\mathbb{R})$$

$$\underset{\substack{\cap \ \ \cap \\ \mathfrak{g} \ \ \mathfrak{t}}}{(A, B)} \longmapsto e^A \cdot e^B$$

$d_0\psi = \mathrm{Id} \Rightarrow \psi$ is a local diffeomorphism around 0

$\Rightarrow \exists\, U, V$ open nbhd of $0 \in M_n$, $\mathrm{Id} \in GL_n(\mathbb{R})$

st. $\psi : U \xrightarrow{\sim} V$

Want to show that we can find U, V small enough such that $\boxed{\psi : U \cap \mathfrak{g} \xrightarrow{\sim} V \cap G}$

$\Rightarrow \psi$ is a local chart around $\mathrm{Id} \in G$

$$\Rightarrow T_{\mathrm{Id}} G = \mathrm{Im}(\underset{\substack{\parallel \\ \mathrm{Id}}}{d_0\psi}) = \mathfrak{g}$$

$\underline{1^{st} \text{ observation}}$ $\quad \psi^{-1}(V \cap G) = U \cap \mathfrak{g}$.

Asme it is not the case $\quad \underset{(A,B)}{\cup} = \mathfrak{g} \oplus \mathfrak{t} \cong M_{n\times n}(\mathbb{R})$

$$\Rightarrow (0, B) \in \psi^{-1}(V \cap G)$$

$$e^A \cdot e^B \in G \iff \underset{\substack{\cap \\ G}}{e^{-A}}\, e^A\, e^B \in G$$

Want to find V small enough such that

$$\psi^{-1}(V \cap G) \cap \mathfrak{t} = \{0\}$$

In other words we want to show that

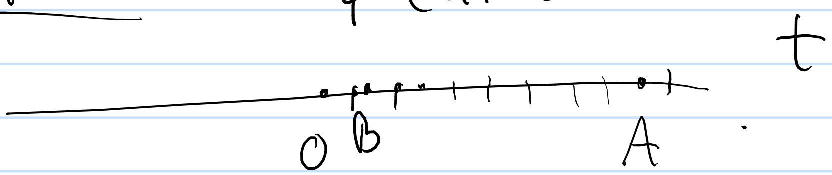
0 is not a limit point of $\psi^{-1}(V \cap G) \cap \mathfrak{t}$

Claim If 0 is a limit point, then there exists a line in $\psi^{-1}(G) \cap \mathfrak{t}$.

$$\left(\text{but } e^{At} \in G \ \forall t \in \mathbb{R} \Rightarrow A \in \mathfrak{g}\right)$$

Lemma 1.2.15

$\psi^{-1}(G) \cap \mathfrak{t}$



$$A \in \psi^{-1}(G) \cap \mathfrak{t} \Rightarrow 2A \in \psi^{-1}(G) \cap \mathfrak{t}$$

$$e^{A} \in G \Rightarrow e^{2A} = (e^{A})^2 \in G$$

Some multiple of B will be very close to A