REPRESENTATIONS OF LIE ALGEBRAS

§ 2.1 - 2.2.

For us Lie group = closed subgroup of $GL_n(\mathbb{R})$

Recall A representation of a Lie group G is a smooth morphism $\rho : G \to GL(V)$ where V f.d. vector space $/\mathbb{R}$

$V \cong \mathbb{R}^n$, $GL(V) \cong GL_n(\mathbb{R})$

$$\mathrm{Lie}\, GL(V) =: \mathfrak{gl}(V) \cong \underset{\underset{M_n(\mathbb{R}).}{\shortparallel}}{\mathfrak{gl}_n(\mathbb{R})}$$

Def A rep. of a Lie algebra $\mathfrak{g}$ is a hom. of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

i.e. ρ is linear. and

$\forall x, y \in \mathfrak{g}$ we have $\rho([x,y]) = [\rho(x), \rho(y)]$

Equivalently
$$\mathfrak{g} \times V \to V$$
$$(x, v) \mapsto x \cdot v$$

$$x \cdot y(v) - y \cdot x(v) = [x,y] \cdot v \quad \forall x, y \in \mathfrak{g} \quad \forall v \in V.$$

Assume G Lie group, $\rho : G \to GL(V)$ rep.
V fin-d. real v. sp.

We can define an action of $\mathfrak{g} := \mathrm{Lie}\, G$ on V?

$X \in \mathfrak{g}$. $\forall t \in \mathbb{R}$ $\quad e^{Xt} \in G$

For $v \in V$

$$\gamma(t) := \rho(e^{Xt})v \in V$$

$$\dot\gamma(0) = \frac{d}{dt}\left(\rho(e^{Xt})v\right)\Big|_{t=0} = \frac{d}{dt}\rho(e^{Xt})\Big|_{t=0} \cdot v$$

$$= d\rho\left(\frac{d}{dt}e^{Xt}\right) \cdot v = d\rho(X).v$$

$$\mathfrak{g} \times V \longrightarrow V$$
$$(X, v) \longmapsto d\rho(X)v$$

Recall

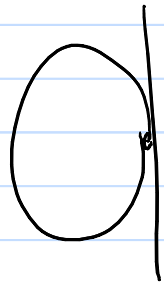
$$\begin{tikzcd}
G \arrow[r, "\rho"] & GL(V) \\
\mathrm{Lie}(G) \arrow[u, "\exp"] \arrow[r, "d\rho"] & \mathfrak{gl}(V) \arrow[u, "\exp"']
\end{tikzcd}$$

ρ group hom
$\Downarrow$
$d\rho$ is Lie algebra hom.

Def Given a rep. ρ of G on V, we call $d\rho$ the derived rep. It is a rep. of $\mathfrak{g}$ on V

Ex. $\rho : S^1 \to GL_2(\mathbb{R})$

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



$$\mathrm{Lie}\, S^1 = T_1 S^1 = i\mathbb{R} \subset \mathbb{C}$$

$$d\rho(i\theta) = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \cos(\theta t) & \sin(\theta t) \\ -\sin(\theta t) & \cos(\theta t) \end{pmatrix}\Big|_{t=0}$$

$$\left\{ \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \right\} \cong \mathfrak{o}_2(\mathbb{R})$$

Ex.2 We defind a rep. of G on $\mathfrak{g}$

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$$
$$(g, X) \mapsto g X g^{-1}$$

What is $d(\mathrm{Ad})$? $X \in \mathfrak{g}$. $Y \in \mathfrak{g}$

$$\frac{d}{dt}\left(e^{Xt} Y e^{-Xt}\right)\Big|_{t=0} = XY - YX = [X, Y]$$

$$d\,\mathrm{Ad} =: \mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$X, Y \mapsto [X, Y].$$

called the adjoint rep. of $\mathfrak{g}$.

Def V_1, V_2 be rep. of $\mathfrak{g}$

$$\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$$
$$\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$$

A morph. of $\mathfrak{g}$-rep. is a linear map $f : V_1 \to V_2$ such that $f \circ \rho_1(X) = \rho_2(X) \circ f \quad \forall X \in \mathfrak{g}$

(i.o.w. $\forall v \in V_1 \; \forall X \in \mathfrak{g}$

$$f(X.v) = X.(f(v)).$$

We denote $\mathrm{Hom}_{\mathfrak{g}}(V_1, V_2) = \{ f : V_1 \to V_2 \text{ morph of } \mathfrak{g}\text{-rep} \}$

$\mathrm{Hom}_G(V_1, V_2) = \{ f : V_1 \to V_2 \text{ morph of } G\text{-rep.} \}$

Thm Assume V_1, V_2 are real representations of a Lie group G. Taking derivative of reps. induces an inclusion

$$\mathrm{Hom}_G(V_1, V_2) \subset \mathrm{Hom}_{\mathfrak{g}}(V_1, V_2)$$

Pf Let $f : V_1 \to V_2$ be a morphism of G-rep.

We want to show $f \in \mathrm{Hom}_{\mathfrak{g}}(V_1, V_2)$.

Let $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$

So $d\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$, $d\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ rep. of $\mathfrak{g}$.

We want to show $\forall X \in \mathfrak{g}$ $\forall v \in V_1$

We want

$$f(d\rho_1(X)v) = d\rho_2(X) f(v)$$

$$\begin{aligned}
&\| \\
&f\left(\frac{d}{dt}(\rho_1(e^{Xt}).v)\right)\Big|_{t=0} = \frac{d}{dt}\left(f(\rho_1(e^{Xt})v)\right)\Big|_{t=0} \\
&\hspace{6cm}\| \\
&\frac{d}{dt}(\rho_2(e^{Xt})\ f(v)) = \frac{d}{dt}(\rho_2(e^{Xt})\ f(v))\Big|_{t=0} \\
&\| \\
&d\rho_2(X)\ f(v)
\end{aligned}$$

$\square$.

APPLICATIONS $\quad V_1 = \mathbb{R}$ trivial rep.

$\forall g \in G \quad g \cdot x = x \quad \forall x \in \mathbb{R}$.

$$\mathrm{Hom}_G(\mathbb{R}, V) \cong V^G := \left\{ v \in V \,\middle|\, \begin{matrix} g \cdot v = v \\ \forall g \in G \end{matrix} \right\}$$

$$f \longmapsto f(1)$$

$$\mathrm{Hom}_G(\mathbb{R}, V) \subset \mathrm{Hom}_{\mathfrak{g}}(\mathbb{R}, V) = V^{\mathfrak{g}} =: \left\{ v \in V \,\middle|\, \begin{matrix} X.v = 0 \\ \forall X \in \mathfrak{g} \end{matrix} \right\}$$

$$V^G \subset V^{\mathfrak{g}}.$$

2) $f: V_1 \xrightarrow{\sim} V_2$ isomorphism of G-rep.

$\iff f: V_1 \xrightarrow{\sim} V_2$ isom. of $\mathfrak{g}$-rep.

$$\mathrm{Hom}_G(V_1, V_2) \subsetneq \mathrm{Hom}_{\mathfrak{g}}(V_1, V_2)$$

Def. If V is a rep. of a Lie alg. $\mathfrak{g}$, then $W \subset V$ is a subrep. is a subvector space which is stable under $\mathfrak{g}$.

- A rep V of $\mathfrak{g}$ is <u>irreducible</u> if $\{0\}$ and V are the only subrep. of $\mathfrak{g}$.

Thm Assume G is a connected Lie group. Let $\rho: G \to GL(V)$ rep. of G. Then $W \subset V$ is a subrep. $\iff$ W subrep. of $\mathfrak{g} := \mathrm{Lie}\, G$ (i.e. W stable under $\mathfrak{g}$).

Pf "$\Longrightarrow$" W rep. of $G \rightsquigarrow W$ rep. of $\mathfrak{g}$. ✓

"$\Longleftarrow$" W subrep. of $\mathfrak{g}$.

<u>Recall</u>

$$\begin{array}{ccc} G & \xrightarrow{\rho} & GL(V) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{d\rho} & \mathfrak{gl}(V) \end{array}$$

$w \in W$ $\quad d\rho(x)\, w \in W \quad \forall x \in \mathfrak{g}.$

$$\exp(d\rho(x))\, w := \sum_{m \geq 0} \underbrace{\frac{(d\rho(x))^m}{m!} w}_{\in W} \in W$$

$$\exp(d\rho(x)) = \rho(e^x) \Rightarrow W \text{ is stable under } \exp(\mathfrak{g})$$

G connected $\Rightarrow$ G is generated by $\exp(\mathfrak{g})$.

$\Rightarrow$ W is stable under G. $\square$

Cor

$$\left\{ \begin{array}{l} \text{irr. f.d.} \\ \text{rep. of } G_{/\mathbb{R}} \end{array} \right\}_{/\simeq} \hookrightarrow \left\{ \begin{array}{l} \text{irr. f.d.} \\ \text{rep. of } \mathrm{Lie}\, G_{/\mathbb{R}} \end{array} \right\}_{/\simeq}$$

Pf

$\rho : G \to GL(V)$ irr. rep.

$d\rho$ is irreducible. $\square$

$$\left\{ \begin{array}{l} \text{irr. f.d.} \\ \text{rep. of } G_{/\mathbb{C}} \end{array} \right\}_{/\simeq} \hookrightarrow \left\{ \begin{array}{l} \text{irr. f.d. rep.} \\ \text{of } (\mathrm{Lie}\, G)^{\mathbb{C}}{}_{/\mathbb{C}} \end{array} \right\}_{/\simeq}$$

$\mathrm{Lie}(G)^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}\, G$ "complexification of a Lie algebra".

REPRESENTATIONS OF $SO_3(\mathbb{R})$ AND $SU_2(\mathbb{C})$. §1.5

$$SU_2(\mathbb{C}) = \left\{ M \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \;\middle|\; \begin{array}{l} \det M = 1 \\ M\bar{M}^t = \mathrm{Id}_2 \end{array} \right\}$$

<u>Lem</u> We have a diffeomorphism
$$S^3 \xrightarrow{\sim} SU_2(\mathbb{C}).$$

std. Hermitean product on $\mathbb{C}^2$

<u>Pf</u>
$$SU_2(\mathbb{C}) = \left\{ M \in M_2(\mathbb{C}) \;\middle|\; \begin{array}{l} \langle Mv, Mw\rangle = \langle v, w\rangle \\ \forall v, w \in \mathbb{C}^2 \\ \det M = 1 \end{array} \right\}$$

$$\Downarrow$$

$$M = \left(v_1 \,\middle|\, v_2 \right) \qquad \begin{array}{l} v_1 \text{ is a vector of } \|v_1\| = 1 \\ \bar{v}_1^t v_2 = 0 \end{array}$$

$$\{ v \in \mathbb{C}^2 \mid \|v\| = 1 \} \cong S^3$$

$$S^3 \cup \{ (a,b,c,d) \mid \textstyle\sum a^2 = 1 \}$$

$$\begin{pmatrix} a+bi \\ c+di \end{pmatrix} \longleftarrow\!\shortmid (a,b,c,d)$$

$$\text{If } v_1 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},\quad v_2 \in \left\langle \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix} \right\rangle \Rightarrow v_2 = \begin{pmatrix} -\lambda\bar{\beta} \\ \lambda\bar{\alpha} \end{pmatrix}$$

for some $\lambda \in \mathbb{C}$

$$\det\begin{pmatrix} \alpha & -\bar{\beta}\lambda \\ \beta & \bar{\alpha}\lambda \end{pmatrix} = \lambda\underbrace{(|\alpha|^2+|\beta|^2)}_{\overset{\shortparallel}{1}} = \lambda$$

$\Rightarrow \lambda = 1$.

So every element in $SU_2(\mathbb{C})$ is of the form $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$

$$S^3 \xrightarrow{\sim} SU_2(\mathbb{C})$$

$$(a,b,c,d) \mapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad \text{where } \alpha = a+bi, \ \beta = c+di.$$

$\square$

What about $SO_3(\mathbb{R})$?

<u>Thm</u> We have Lie group hom.

$$SU_2(\mathbb{C}) \to SO_3(\mathbb{R}) \text{ with } \{\pm \mathrm{Id}\}$$

<u>Pf</u> $\mathrm{Lie}\, SU_2(\mathbb{C}) =: su_2(\mathbb{C})$

$$\shortparallel$$

$$\{A \in M_2(\mathbb{C}) \mid e^{tA} \in SU_2(\mathbb{C}) \ \forall t \in \mathbb{R}\}$$

$$\shortparallel$$

$$\left\{A \in M_2(\mathbb{C}) \;\middle|\; \begin{array}{l} \mathrm{tr}\, A = 0 \\ A + \bar{A}^t = 0 \end{array}\right\}$$

$$\mathfrak{su}_2(\mathbb{C}) = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

$$\mathrm{Ad} : SU_2(\mathbb{C}) \longrightarrow GL(\mathfrak{su}_2(\mathbb{C})) \cong GL_3(\mathbb{R}).$$

$$\ker(\mathrm{Ad}) = \left\{ g \in SU_2(\mathbb{C}) \;\middle|\; gA = Ag \quad \forall A \in \mathfrak{su}_2(\mathbb{C}) \right\}$$

$$" \{\pm \mathrm{id}\}$$

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad gA = Ag \Rightarrow g \text{ diagonal}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} q & 0 \\ 0 & \bar{q} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q & 0 \\ 0 & \bar{q} \end{pmatrix}$$

$$\Rightarrow q^2 = 1 \Rightarrow q = \pm 1$$

$$\mathrm{Im}(\mathrm{Ad}) \cong SU_2(\mathbb{C}) / \{\pm \mathrm{Id}\}.$$

We define a scalar product on $\mathfrak{su}_2(\mathbb{C})$ by

$$\langle A, B \rangle = -\mathrm{tr}(AB)$$

It's positive definite.

$$-\mathrm{tr}\left(\begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \begin{pmatrix} ia' & b'+ic' \\ -b'+ic' & -ia' \end{pmatrix} \right) =$$

$$-\mathrm{tr} \begin{pmatrix} -aa'-bb'-cc' & \\ & -aa'-bb'-cc' \end{pmatrix}$$

$SU_2(\mathbb{C})$ preserve this scalar product.

$$g \in SU_2(\mathbb{C}) \qquad \langle gAg^{-1}, gBg^{-1}\rangle = -\mathrm{tr}(gABg^{-1})$$

$$\langle gAg^{-1}, gBg^{-1}\rangle = \langle A, B\rangle = -\mathrm{tr}(AB) = -\mathrm{tr}(gABg^{-1})$$

$$\Rightarrow \mathrm{Im}(\mathrm{Ad}) \subset O_3(\mathbb{R})$$

$SU_2(\mathbb{C})$ conected $\Rightarrow \mathrm{Im}(\mathrm{Ad}) \subset SO_3(\mathbb{R})$.

$$\begin{tikzcd}
U \ni 1 \arrow[rrr, bend left] & SU_2(\mathbb{C}) \arrow[r, "\rho"] & SO_3(\mathbb{R}) & \\
0 \arrow[u, "S"] \arrow[rrr, bend right] & \mathfrak{su}_2(\mathbb{C}) \arrow[u, "\exp"] \arrow[r, "\cong"] & \mathfrak{so}_3(\mathbb{R}) \arrow[u, "\exp"'] &
\end{tikzcd}$$

3-dim

$$SU_2(\mathbb{C})/\{\pm \mathrm{Id}\} \cong \mathrm{Im}(\mathrm{Ad}) = SO_3(\mathbb{R})$$

Spin group.

3D Rotation group.