HAAR MEASURE §2.4–2.5.

SEPANSKI's Book §1.4.

M, N differentiable mflds

$\Phi : M \to N$ differentiable.

induces $\quad d_p\Phi : T_pM \to T_{\Phi(p)}N$

If $\gamma(t) \in M$ $\mathcal{C}^\infty$ curve,

$$d_p\Phi(\dot\gamma(0)) := \frac{d}{dt}\Phi(\gamma(t))\Big|_{t=0}$$

Take dual spaces $\quad T_p^*M = (T_pM)^*$

so we also get

$$\Phi^* : T^*_{\Phi(p)}N \to T_p^*M$$

Assume $\dim M = \dim N = n$

also get $\quad \Phi^* : \Lambda^n T^*_{\Phi(p)}N \to \Lambda^n T_p^*M$

<u>EXAMPLE</u> If $\varphi : U \xrightarrow{\sim} V$ at $p \in M$, $U \subseteq \mathbb{R}^n$, $V \subseteq M$

$$\varphi^* : \Lambda^n T_p^*M \xrightarrow{\sim} \Lambda^n T^*_{\varphi^{-1}(p)}\mathbb{R}^n = \mathbb{R}\, dx_1 \wedge \dots \wedge dx_n$$

where $x_1, \dots, x_n$ basis of $\mathbb{R}^n$.

Def A volume form is a map

linear bundle on M.

$$\omega : M \longrightarrow \coprod_{p \in M} \Lambda^m T_p^* M$$

where $\forall p \in M$ $0 \neq \omega(p) \in \Lambda^m T_p^* M$

such that it is differentiable.

Differentiable means that $\forall\ \varphi : U \xrightarrow{\sim} V$ chart

$$\begin{tikzcd}
V \arrow[r, "\omega"] \arrow[drr, bend right] & \coprod_{p \in V} \Lambda^m T_p^* M \arrow[r, "\varphi^*_{\sim}"] & \coprod_{p \in V} \Lambda^m T^*_{\varphi^{-1}(p)} \mathbb{R}^m \\
 & & V \times \mathbb{R}
\end{tikzcd}$$

$c\,dx_1 \wedge \dots \wedge dx_m \in \Lambda^m T^*_{\varphi^{-1}(v)} \mathbb{R}^m$

$\| \quad \uparrow$

(v, c)

we ask that this map is diff.

How to integrate? $\qquad \varphi : U \xrightarrow{\sim} V$ chart.

$f : M \to \mathbb{R}$ continuous. f is supported on a chart V as before

$\operatorname{supp} f = \{x \in M \mid f(x) \neq 0\}$

$$\int_M f = \int_M f\,\omega_M = \int_U f\ \underbrace{\varphi^* \omega}_{g\,dx_1 \wedge \dots \wedge dx_m}$$

(We need to restrict to chart with respect to which $g > 0$)

(Can remove the condition that f supp. on V by using a partition of 1)

INVARIANT FORMS ON LIE GROUPS.

G Lie group, $g \in G$ $\qquad \dim G = n$

$$l_g : G \to G, \qquad r_g : G \to G$$
$$h \mapsto gh \qquad\qquad h \to hg$$

<u>Def</u> A volume form ω on G is said left-invariant if $l_g^* \omega = \omega$ $\forall g \in G$ (right-inv. if $r_g^* \omega = \omega$ $\forall g \in G$)

<u>Lem</u> Up to mult. by a scalar, there exists a unique left invariant volume form on G.

<u>Pf</u> $\dim T_e^* G = n$, $\dim \Lambda^n T_e^* G = 1$

$$\underset{\omega_e}{\overset{\Psi}{\Lambda^n T_e^* G}}$$

We can extend it to a global form by setting

$$\omega_g := l_{g^{-1}}^* \omega_e \qquad dl_{g^{-1}} : T_g G \to T_e G$$
$$l_{g^{-1}}^* : \Lambda^n T_e^* G \to \Lambda^n T_g^* G$$
$$\omega_e \qquad\qquad l_{g^{-1}}^*(\omega_e)$$

Let's check that ω is left invariant.

$\forall h \in G$ $\quad l_h^* \omega = \omega$.

$$l_h^* \omega = \omega \iff \forall g \in G$$

$$(l_h^* \omega)_g = \omega_g \qquad dl_h : T_g G \to T_{hg} G$$

$$T_e G \xrightarrow{l^*_{g^{-1}h^{-1}}} T^*_{hg} G \xrightarrow{l_h^*} T_g^* G$$

$$(l_h^* \omega)_g = l_h^* \omega_{hg} = l_h^* \, l^*_{g^{-1}h^{-1}} \omega_e =$$

$$= (l_{g^{-1}h^{-1}} \circ l_h)^* \omega_e =$$

$$= l^*_{g^{-1}} \omega_e = \omega_g . \qquad \square$$

Lemma If G is compact, then there exists a unique volume form ω (up to mult. ± 1) such

$$\int_G 1 = 1 .$$

Pf. $\int_G \omega < \infty$, so we can rescale by some $C \in \mathbb{R}_{>0}$ to obtain $\int_G \omega = 1$.

This ω is unique up to ± 1

If G is compact we can the integration on G as
$f: G \to \mathbb{R}$ continuous

$$\int_g f\, dg := \int_G f\omega \quad , \text{where } \int_G \omega = 1 \text{ as in the Lem.}$$

dg is called the Haar measure on G.

<u>Lem</u> dg is right invariant.

<u>Pf</u> $\forall g, h$ $\quad l_g, r_h$ commute.

so $r_h^* \omega$ is left invariant.

$$r_h^* \omega = c(h)^{-1} \omega \quad , \quad c(h) \in \mathbb{R} \setminus \{0\}$$

$$r_{gh}^* = r_h^* \circ r_g^* \quad \forall g, h \Rightarrow c: G \to \mathbb{R} \setminus \{0\}$$
is group hom.

G is compact $\Rightarrow$ $c(G)$ is compact.

$$\Rightarrow c(G) \subset \{-1, +1\}$$

so $\quad r_h^* \omega = \pm \omega$

$\rightsquigarrow$ so in any case $\int r_h^* \omega = \int \omega$

so dg is right invariant.

$\forall h \ \forall f: G \to \mathbb{R}$

$$\int_G f(hg)\,dg = \int_G f(g)\,dg = \int_G f(gh)\,dg$$

$$\| $$

$$\int_G (f \circ \ell_h)(g)\,dg$$

$$\|$$

$$\int_G (f \circ \ell_h)(g)\,\omega = \int_G f(g) \underset{\omega}{\underset{\|}{(\ell_h^* \omega)}} = \int_G f(g)\,\omega$$

EXAMPLES OF HAAR MEASURES.

1) $(\mathbb{R},+)$ $\quad dg = dx$ usual Lebesgue measure

$$\omega: \mathbb{R} \to \coprod T_p^* \mathbb{R}. \ni \sigma: T_p\mathbb{R} \xrightarrow{\sim} \mathbb{R}$$

$$p \longmapsto (\underset{T_p\mathbb{R} \cong \mathbb{R}.}{\underset{\cap}{v}} \mapsto v)$$

$$x \in \mathbb{R}.\ (\ell_x^* \omega)_y(\dot{\gamma}(0)) = \omega_{xy}\left(\frac{d}{dt}(x+\gamma(t))\right)_{t=0} =$$

$$\| \qquad = \omega_{xy}(\dot{\gamma}(0)) = \dot{\gamma}(0)$$

$$\omega_y(\dot{\gamma}(0)) = \dot{\gamma}(0).$$

$\underline{2)}$ $G = (\mathbb{R}^*, \cdot)$ $\qquad dg = \dfrac{dx}{|x|}$

this comes from the volume form.

$$\omega := \frac{dx}{x} : \mathbb{R}^\times \longrightarrow \bigsqcup T_p^* \mathbb{R}^*$$

$$p \longrightarrow \left(\dot\gamma(0) \to \frac{\dot\gamma(0)}{p} \right)$$

$y \in \mathbb{R}^*$.
$$\left(l_y^* \frac{dx}{x}\right)_z (\dot\gamma(0)) = \left(\frac{dx}{x}\right)_{yz} \left(\frac{d}{dt} (y\gamma(t)) \Big|_{t=0} \right)$$

$$= \left(\frac{dx}{x}\right)_{yz} (y\,\dot\gamma(0)) = \frac{\cancel{y}\,\dot\gamma(0)}{\cancel{y} z}$$

$$= \left(\frac{dx}{x}\right)_z (\dot\gamma(0))$$

$f : \mathbb{R}^* \to \mathbb{R}$ continuous

$$\int_{\mathbb{R}^*} f \, dg = \int_{\mathbb{R}^*} f(x) \frac{dx}{|x|} .$$

$A \subset \mathbb{R}^*$ $\qquad \mu(A) = \displaystyle\int_A \frac{dx}{|x|} .$ $\qquad \mu(yA) = \mu(A)$ $\forall y \in \mathbb{R}^*$.

·) $S^1 = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \cong \mathbb{R}/2\pi\mathbb{Z}$.

$d\theta$ is the left invariant form.

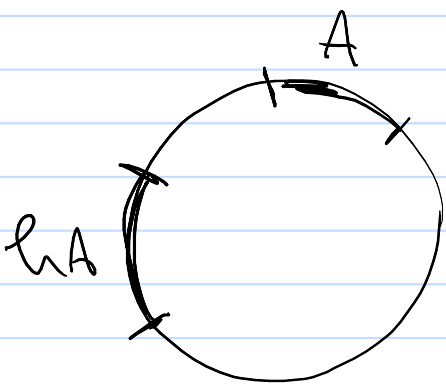
$\gamma(t) \in S^1$ we can write $\gamma(t) = e^{i\delta(t)}$

$d\theta(\dot\gamma(0)) = \dot\delta(0)$

$g \in S^1$. $(l_g^* d\theta)_h (\dot\gamma(0)) = (d\theta)_{gh}\left(\frac{d}{dt} g\gamma(t)\Big|_{t=0}\right)$

$\overset{\shortparallel}{e^{i\varphi}}$ $\quad (d\theta)_{gh}\left(\frac{d}{dt} e^{i(\varphi+\delta(t))}\Big|_{t=0}\right) =$

$= \dot\delta(0)$.



• $G = GL_n(\mathbb{R}) \subset M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$

let ω_M the usual volume form on $M_{n\times n}(\mathbb{R})$

$\omega_M = de_{11} \wedge de_{12} \wedge \dots \wedge de_{nn}$.

$A \in GL_n(\mathbb{R})$

$$\omega_G = \frac{\omega_M}{(\det A)^n}.$$

$$\ell_B : M_{m\times m}(\mathbb{R}) \longrightarrow M_{m\times m}(\mathbb{R})$$
$$C \longmapsto BC$$

$$\det(\ell_B) = (\det B)^m.$$

$$(\ell_B^* \omega_G)_A(e_{11},\dots,e_{mm}) = (\omega_G)_A(e_{11},\dots,e_{mm}).$$

$$\Lambda^{m^2} T_A^* M_{m\times m}(\mathbb{R}) = \Big\{ f : \underbrace{(T_p M_{m\times m}(\mathbb{R}))^m \to \mathbb{R}}_{\text{multilinear}} \;\Big|\; \begin{array}{l} f(v_1,\dots,v_{m^2})=0 \\ \text{if } v_i = v_j \\ \text{for } i\neq j \end{array} \Big\}$$

$$(\omega_G)_{BA}(Be_{11}, Be_{12},\dots,Be_{mm}) = \frac{\omega_M(Be_{11},\dots,Be_{mm})}{(\det B)^m(\det A)^m} = \frac{\cancel{(\det B)^m}\,\omega_M(e_{11},\dots,e_{mm})}{\cancel{(\det B)^m}(\det A)^m}$$

APPLICATION OF HAAR MEASURES.

Lemma If V is a real f.d. rep. of a compact Lie group G. Then there exists a G-inv. scalar product on V.

G-inv. means $(gv, gw) = (v,w)$ $\forall g\in G$ $\forall v,w\in V$.

Pf We start with an arbitrary scalar product $b(-,-)$ on V.

$$(v,w) := \int_G b(gv, gw)\, dg.$$

$(-,-)$ is invariant scalar product.

- $(-,-)$ bilinear since b is bilinear

- if $v \neq 0$, $(v,v) > 0$ because $b(gv, gv) > 0 \ \forall g$.

$(-,-)$ is G-inv: $h \in G$

$$(hv, hw) = \int_G b(ghv, ghw)\, dg = \int_G b(gv, gw)\, dg = (v,w)$$

$$\boxed{\int_G f(g)\, dg = \int_G f(gh)\, dg}$$

$\square$

Thm Every f.d. rep. of a compact Lie group G is a direct sum of irred. rep.

Pf V rep. of G. If V irred. ✓.

If not $\exists\, W \subset V$ G-subrep.

$\exists\, (-,-)$ G-inv. scalar product.

$$V = W \oplus W^\perp$$

$W^\perp$ is a rep. of G

$$x \in W^\perp, \quad (gx, w) = (x, \underset{\substack{\cap \\ W}}{g^{-1}w}) = 0 \Rightarrow gx \in W^\perp$$
$w \in W$

$\Rightarrow V = W \oplus W^\perp$. Since $\dim W, \dim W^\perp$ can conclude by induction $\square$

Irreducible rep. of a group G are called simple rep.

A rep. which is direct sum of irred. rep. is called semisimple (or completely reducible).

$V = V_1 \oplus \ldots \oplus V_n$. Is this unique?

irreducible

Lem V, W irr. rep. of a group G.

Assume $f : V \to W$ homomorphism of rep.

Then either $f = 0$ or f = iso.

Pf. f hom. $\Rightarrow \ker f, \operatorname{Im} f$ are rep. of G.

$\ker f = 0$ or $\ker f = V$, $\Rightarrow f = 0$ or f injective

$\operatorname{Im} f = 0$ or $\operatorname{Im} f = W \Rightarrow f = 0$ or f surjective

If $f \neq 0$, f is isomorphism.

<u>Schur Lemma</u> If V complex irr. of a group G.

then $\mathrm{End}_G(V) \cong \mathbb{C}\,\mathrm{Id}_V$.

<u>Pf.</u> $f: V \to V$ is a G-hom.

$\forall \lambda \in \mathbb{C}$, $f - \lambda\,\mathrm{Id}$ is a G-hom.

Let λ be an eigenvalue of f.

$\ker(f - \lambda\,\mathrm{Id}) \neq 0 \Rightarrow \ker(f - \lambda\,\mathrm{Id}) = V$

$\Rightarrow f - \lambda\,\mathrm{Id} = 0 \Rightarrow f = \lambda\,\mathrm{Id}$.

If V, W complex irr. rep. of a group G.

$$\dim \mathrm{Hom}_G(V,W) = \begin{cases} 0 & \text{if } V \not\cong W \\ 1 & \text{if } V \cong W. \end{cases}$$

<u>Prop</u> Let V be complex f.d rep. of G

and assume $V = V_1 \oplus \dots \oplus V_m = V_1' \oplus \dots \oplus V_{m'}'$.

then $m = m'$, and $\exists \sigma \in S_m$ s.t.

$\forall i \quad V_i \cong V'_{\sigma(i)}$.

Pf If L is irr. rep. of G

$$\dim \mathrm{Hom}_G(L, V) \cong \dim \mathrm{Hom}_G(L, \bigoplus V_i) =$$

$$\cong \dim\Big(\bigoplus_i \mathrm{Hom}_G(L, V_i)\Big) = \sum_i \dim \mathrm{Hom}_G(L, V_i)$$

$$= \#\{i \mid V_i \cong L\}$$

$$= \#\{j \mid V'_j \cong L\}$$

$\square$