PETER-WEYL THEOREM
& SPHERICAL HARMONICS §2.6.

Last time: MASCHKE THEOREM

Every fin. dimensional rep. of a compact Lie group is irreducible

$S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ $\qquad S^1 \circlearrowleft S^1.$

$$L^2(S^1) = \left\{ f : S^1 \to \mathbb{C} \text{ measurable such that } \frac{1}{2\pi}\int_{S^1} |f(\theta)|^2 d\theta < \infty \right\}$$

$L^2(S^1)$ is a rep. of the group S^1
&
$g \cdot f(z) = f(g^{-1}z).$

χ_n the rep. of S^1 on $\mathbb{C}$ given by $z \to z^n$

$\varphi \in \mathrm{Hom}_{S^1}(\chi_n, L^2(S^1))$?

$\forall v \in \mathbb{C}, z \in S^1$

$$\begin{aligned} e^{i\theta} \cdot \varphi(v)(z) &= \varphi(v)(e^{-i\theta} z) \overset{z=1}{\Rightarrow} \\ \| \\ \varphi(e^{in\theta} v)(z) &= e^{in\theta} \varphi(v)(z) \end{aligned}$$

$$\varphi(v)(e^{-i\theta}) = e^{in\theta}\varphi(v)(1).$$

$$\Rightarrow \varphi(v)(e^{-i\theta}) = c\, e^{in\theta}$$

$$\dim \mathrm{Hom}_{S^1}(\chi_{n}, L^2(S^1)) = 1$$

$$1 \mapsto (e^{i\theta} \to e^{-in\theta})$$

$$V := \bigoplus_{n\in\mathbb{Z}} \langle e^{in\theta}\rangle \subset L^2(S^1).$$

this space V is dense $\Leftrightarrow V^{\perp} = 0$

$L^2(S^1)$ is a Hilbert space : we have a scalar product $\langle f,g\rangle = \frac{1}{2\pi}\int_{S^1} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta$

this is invariant under S^1.

$$\forall n \quad \int_{S^1} f(e^{i\theta})\, e^{-in\theta}\, d\theta = 0 \Rightarrow f = 0$$

$$f \in L^2(S^1) \Rightarrow f = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

where $c_n = \frac{1}{2\pi}\int f(e^{i\theta})\, e^{-in\theta}\, d\theta.$

PETER-WEYL THEOREM

G compact group. V_ξ an irred. rep. of G.

$L^2(G)$ is a rep. of G.

$$\dim \operatorname{Hom}_G(V_\xi, L^2(G)) = \dim V_\xi .$$

$L^2(G)_\xi$ be the isotopic component of $L^2(G)$

$$L^2(G)_\xi = \sum \{ W \subset L^2(G) \mid \underset{\text{as a subrep.}}{W \cong V_\xi} \}$$

$$= \operatorname{span}(\operatorname{Im}(\varphi) \mid \varphi \in \operatorname{Hom}_G(V_\xi, L^2(G)) > .$$

$$\bigoplus_{\xi \text{ irr.}} L^2(G)_\xi \subset L^2(G) \text{ which is dense.}$$

$\rightsquigarrow$ NON-ABELIAN FOURIER ANALYSIS.

THE CASE OF $SO_3(\mathbb{R})$ ACTING ON S^2.

$SO_3(\mathbb{R}) \curvearrowright \mathbb{R}^3$ preserves $S^2 \subset \mathbb{R}^3$.

So we get an action of $SO_3(\mathbb{R}) \curvearrowright L^2(S^2)$.

$L^2(S^2)$ is Hilbert space

$$\langle f,g\rangle := \int_{S^2} f(x)\overline{g(x)}\,dx.$$

This scalar product is $SO_3(\mathbb{R})$-invariant.

$y \in SO_3(\mathbb{R})$.

$$z = y^{-1}x$$

$$\langle yf, yg\rangle = \int_{S^2} f(y^{-1}x)\,\overline{g(y^{-1}x)}\,dx =$$

$$\int_{S^2} f(z)\,\overline{g(z)}\,d(yz) = \underset{\langle f,g\rangle.}{\underset{\shortparallel}{\int_{S^2} f(z)\overline{g(z)}\,\overset{\overset{1}{\shortparallel}}{|\ell_y|}\,dz}}$$

Recall

$$\left\{\begin{array}{l}\text{in. rep. of}\\ SO_3(\mathbb{R})\ /\mathbb{C}\end{array}\right\}/\cong \quad\longleftrightarrow\quad \{1,3,5,\dots\}$$

$$L(2\ell) \longrightarrow 2\ell+1$$

$$\dim L(2\ell) = 2\ell+1.$$

Thm $\cdot)$ $\dim \operatorname{Hom}_{SO_3(\mathbb{R})}\left(L(2\ell), L^2(S^2)\right) = 1 \ \forall \ell \in \mathbb{N}$

$\cdot)$ sum of all the irr. subrep. of $L^2(S^2)$ is a dense subspace of $L^2(S^2)$.

Pf $L^2(S^2) \supset \mathcal{C}(S^2) = \{\text{continuous fct. } S^2 \to \mathbb{C}\}$

$\mathcal{C}(S^2)$ are dense in $L^2(S^2)$.

$\mathbb{C}[X,Y,Z]^\ell$ hom. polynomials of degree ℓ

$$\Phi_\ell : \mathbb{C}[X,Y,Z]^\ell \hookrightarrow \mathcal{C}(S^2)$$
$$P \longmapsto \left((x,y,z) \mapsto P(x,y,z)\right)$$

$$\mathcal{C}^\ell = \operatorname{Im} \Phi_\ell$$

ℓ even $\Rightarrow \mathcal{C}^\ell \subset \mathcal{C}(S^2)^+ = \{f \mid f(x) = f(-x) \ \forall x \in S^2\}$

ℓ odd $\Rightarrow \mathcal{C}^\ell \subset \mathcal{C}(S^2)^- = \{f \mid f(x) = -f(-x) \ \forall x \in S^2\}$

Claim $\mathcal{C}^\ell \subset \mathcal{C}^{\ell+2} \ \forall \ \ell \geq 0$.

Pf of claim $X^2 + Y^2 + Z^2 \big|_{S^2} \equiv 1$

$f \in \mathcal{C}^\ell$, then $f = \Phi_\ell(P)$.

$$f = \Phi_{\ell+2}(P(x^2+y^2+z^2))$$

$\Rightarrow f \in \mathcal{C}_{\ell+2}$.

$$\mathcal{C}^0 \subset \mathcal{C}^2 \subset \dots \subset \mathcal{C}^{2n} \subset \dots \subset \mathcal{C}(S^2)^+$$

$$\mathcal{C}^1 \subset \mathcal{C}^3 \subset \dots \subset \mathcal{C}^{2n+1} \subset \dots \subset \mathcal{C}(S^2)^-$$

Chains of $SO_3(\mathbb{R})$-representations

Each $\mathcal{C}^\ell$ is a subrep. of $SO_3(\mathbb{R})$.

$P \in \mathcal{C}^\ell$.

$$g^{-1} = \begin{pmatrix} a_{11} & a_{21} & \cdots \\ a_{12} & & \ddots \\ & & \end{pmatrix}$$

$$g \cdot P(x,y,z) = P(g^{-1}(x,y,z)) =$$

$$= P(a_{11}x + a_{21}y + a_{31}z, a_{12}x + \dots, \dots)$$

$\Rightarrow g \cdot P \in \mathcal{C}^\ell$

$\mathcal{C}^\ell \subset \mathcal{C}^{\ell+2}$. Let $\mathcal{H}^{\ell+2}$ be the orthogonal of $\mathcal{C}^\ell$ in $\mathcal{C}^{\ell+2}$.

$\mathcal{H}^{\ell+2}$ is a rep. of $SO_3(\mathbb{R})$.

$$\dim \mathcal{H}^\ell = \dim \mathcal{C}^\ell - \dim \mathcal{C}^{\ell-2}.$$

$$\dim \mathcal{C}^\ell = \dim \mathbb{C}[x,y,z]^\ell = \binom{\ell+2}{2} = \frac{(\ell+2)(\ell+1)}{2}$$

$$1 \quad 2 \;\Big|\; \cdots \;\Big|\; \ell-1 \quad \ell$$

$$\dim \mathcal{H}^\ell = \frac{(\ell+2)(\ell+1) - \ell(\ell-1)}{2} = \frac{4\ell+2}{2} = 2\ell+1$$

$\mathcal{H}^\ell$ is a good candidate for $L(2\ell)$.

<u>Claim</u> $\mathcal{H}^\ell$ is irreducible.

<u>Pf.</u> $S^1_z \subset SO_3(\mathbb{R})$ rotations along z axis

$$\begin{array}{l} S^1_z \\ \| \\ \{A_\theta\} \text{ where } A_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$S^1_z \cong S^1$ as Lie groups.

How does $\mathcal{C}^\ell$ decompose as a rep. of S^1_z?

$$\begin{array}{c} \mathcal{C}^\ell \\ \wr\| \\ \mathbb{C}[x,y,z]^\ell \\ \rotatebox{90}{$\in$} \\ P \end{array} = \bigoplus_{k=0}^{\ell} \mathbb{C}[x,y]^k z^{\ell-k}$$

$$A_\theta \cdot P(x,y,z) = P(\cos\theta\, x - \sin\theta\, y,\ \sin\theta\, x + \cos\theta\, y,\ z)$$

What is $\mathbb{C}[X,Y]^k$ as a rep. of S^1_z?

$\uparrow$

dim $= k+1$.

$$\mathbb{C}[X,Y]^k \hookrightarrow \mathbb{C}[X,Y]^{k+2}$$
$$P \longmapsto P(X^2+Y^2)$$

so $\mathbb{C}[X,Y]^{k+2} \cong \mathbb{C}[X,Y]^k \oplus V_{k+2}$

where V_{k+2} rep. of dim 2.

$$A_\theta \cdot (x+iy)^{k+2} = (\cos\theta\, x - \sin\theta\, y + i(\cos\theta\, y + \sin\theta\, x))^{k+2}$$
$$= (e^{i\theta}x + ie^{i\theta}y)^{k+2} = e^{i(k+2)\theta}(x+iy)$$

so $\langle (x+iy)^{k+2} \rangle \cong \chi_{k+2}$

$\langle (x-iy)^{k+2} \rangle \cong \chi_{-k+2}$.

Claim $\mathbb{C}[X,Y]^k \cong \bigoplus_{i=0}^{k} \chi_{-k+2i}$

Pf. clear if $k=0$,

if $k=1$ $\mathbb{C}[X,Y] \cong \mathbb{C}(x+iy) \oplus \mathbb{C}(x-iy)$

$\cong \chi_1 \oplus \chi_{-1}$.

if $k \geq 2$ $\mathbb{C}[X,Y]^{k+2} \cong \mathbb{C}[X,Y]^k \oplus V_{k+2}$

$\Rightarrow$ by induction $V_{k+2} \cong \chi_{k+2} \oplus \chi_{-k-2}$

$$\Rightarrow \mathbb{C}[x,y]^k \cong \bigoplus_{i=0}^{k} \chi_{-k+2i}.$$

$$\mathbb{C}[X,Y,Z]^{\ell} \cong \bigoplus_{k=0}^{\ell} \mathbb{C}[X,Y]^k Z^{\ell-k} \cong$$

$$\cong \bigoplus_{k=0}^{\ell} \left(\bigoplus_{i=0}^{k} \chi_{-k+2i} \right)$$ as an rep. of S^1_Z.

all the χ_i in $\mathbb{C}[X,Y,Z]^{\ell}$ go from $\chi_{-\ell}$ to χ_{ℓ}.

In particular $\chi_{\ell+2}$ does not occur $\mathbb{C}[X,Y,Z]^{\ell}$!

but it occurs in $\mathcal{C}^{\ell+2}$ – $(x+iy)^{\ell+2}$

Claim $\mathbb{C}[X,Y,Z]^{\ell} \cong \bigoplus_{k=0}^{\lfloor \ell/2 \rfloor} L(2\ell-4k)$

Pf of claim By induction. $\ell=0$ both sides are trivial

$\ell=1$ " " are natural

$\ell \geq 2$. $\mathbb{C}[X,Y,Z]^{\ell} \cong \mathbb{C}[X,Y,Z]^{\ell-2} \oplus \mathcal{H}^{\ell}$

$$\cong \bigoplus_{k=0}^{\lfloor \frac{\ell}{2} \rfloor -1} L(2(\ell-2)-4k)$$

to conclude need to show $\mathcal{H}^{\ell} \cong L(2\ell)$.

as a rep. of S^1_Z χ_{ℓ} occurs in $\mathbb{C}[X,Y,Z]^{\ell}$ but not $\mathbb{C}[X,Y,Z]^{\ell-2}$.

$\Longrightarrow$ χ_ℓ must occur in $\mathcal{H}^\ell$.

So $\mathcal{H}^\ell$ is a rep. which is not isomorphic to any rep. of $\mathbb{C}[X,Y,Z]^h$, for $h<\ell$.

$\Rightarrow$ $\mathcal{H}^\ell \simeq L(2\ell)$, because

$\dim L(2h) > \dim \mathcal{H}^\ell$ for $h>\ell$.

$\Rightarrow$ $\mathcal{H}^\ell$ is irreducible.

So $\dim \operatorname{Hom}_{SO_3(\mathbb{R})}(L(2\ell), L^2(S^2)) \geq 1$

$\mathrm{Pol}(S^2) \subset \mathcal{C}(S^2) \subset L^2(S^2)$

dense subspace

dense from Stone-Weierstrass thm.

$\| \bigcup_{\ell \geq 0} \mathcal{E}^\ell$

$\nwarrow$ Here there exists $\exists!$ copy of $L(2\ell)$.

$\forall \ell$ $\quad \operatorname{Hom}_{SO_3(\mathbb{R})}(L(2\ell), \mathrm{Pol}(S^2)) \overset{\sim}{=} \operatorname{Hom}_{SO_3(\mathbb{R})}(L(2\ell), L^2(S^2))$

$$\mathrm{Pol}(S^2) \simeq \bigoplus_{\ell=0}^{\infty} \mathcal{H}^\ell \quad \Bigg) \quad \overset{\text{|| Schur's lemma}}{\operatorname{Hom}_{SO_3(\mathbb{R})}}(L(2\ell), \mathcal{H}^\ell).$$

$\mathcal{H}^\ell \simeq L(2\ell)$

$\Rightarrow$ $\dim \operatorname{Hom}_{SO_3(\mathbb{R})}(L(2\ell), L^2(S^3)) = 1$.

SPHERICAL HARMONICS

Want to find Hilbert basis of $L^2(S^2)$

(in the same way as $\{e^{in\theta}\}$ was Hilbert basis of $L^2(S^1)$)

SCHUR'S LEMMA then rep. $\mathcal{H}^\ell \subset L^2(S^2)$ are orthogonal to each other.

$\Rightarrow$ it is enough to find an orth. basis of each of the $\mathcal{H}^\ell$.

$S^1_z \subset SO_3(\mathbb{R})$

$\dim (\mathcal{H}^\ell)^{S^1_z} = 1$ (because $\mathcal{H}^\ell = \bigoplus_{k=0}^{\ell} \chi_{-\ell+2k}$)

$\|$

$\{h \in \mathcal{H}^\ell \mid A_\theta \cdot h = h \ \forall \theta \in \mathbb{R}\}$

All the polynomial in $\mathbb{C}[z]$ define invariant fcts wrt S^1_z.

$P \in \mathbb{C}[z]$ of degree ℓ $\Rightarrow P \in L^2(S^2)^{S^1_z}$

and $P \in \sum_{k \leq \ell} \mathcal{P}^k = \bigoplus_{k=0}^{\ell} \mathcal{H}^k$

$\Rightarrow P \in \bigoplus_{k=0}^{\ell} (\mathcal{H}^k)^{S^1_z}$

We want to find an element in $(\mathcal{H}^\ell)^{S^1_z}$

it is enough to find an element orthogonal to $(\mathcal{H}^k)^{S^1_z}$, $k<\ell$.

$$\overline{(\mathcal{H}^0)^{S^1_z} \ni 1 =: P_0(z)}$$

$$\underset{\underset{\mathbb{C}[x,y,z]^1}{\wr\wr}}{(\mathcal{H}^1)^{S^1_z}} \ni z =: P_1(z).$$

$P_2(z) \in \mathbb{C}[z]$ of degree 2, must be orthogonal to 1 and z.

$P_2(z) = a + bz^2$, it is orthogonal to $P_1(z) = z$

$$\langle P_2(z), P_0(z) \rangle = 0 \Rightarrow$$

$$0 = \int_{S^2} P_2(z)\, d\mu = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a + b\sin^2\theta) \cos\theta \, d\theta \, d\varphi$$

$$= 2\pi\left(2a + 2\frac{b}{3}\right) \Rightarrow b = -3a$$

$$P_2(z) = c(3z^2 - 1).$$

$\vdots$

$P_\ell(z) \in (\mathcal{H}^\ell)^{S^1_z}$ ← LEGENDRE POLYNOMIALS

$$P_\ell(z) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dz}\right)^\ell (z^2-1)^\ell$$

How do we get a basis of $\mathcal{H}^\ell$?

$\mathcal{H}^\ell$ is a complex rep. of $SO_3(\mathbb{R})$

$\Rightarrow \mathcal{H}^\ell$ is a rep. of $\mathfrak{so}_3(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$

$E_1 = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$ $\qquad 2iE_3 \mapsfrom h$

$E_2 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}$ $\qquad E_2 - iE_1 \mapsfrom e$

$\qquad -E_2 - iE_1 \mapsfrom f$

$E_3 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$

If $P_\ell(z) \in \mathbb{C}[z]$ so $h \cdot P_\ell(z) = 0$

$P_\ell(z) \in (\mathcal{H}^\ell)_0$

so a basis of $\mathcal{H}^\ell$ is $\left\{ P_\ell(z), e^m P_\ell(z), f^m P_\ell(z) \,\middle|\, 1 \le m \le \ell \right\}$

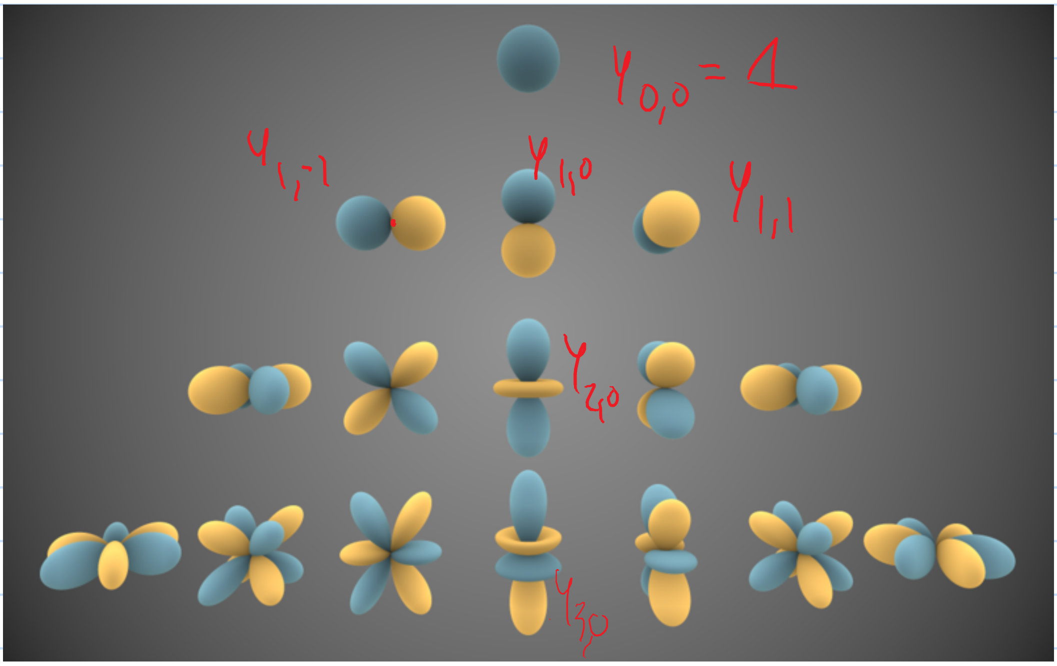
$\parallel$

$Y_{\ell,m}$

CALLED "SPHERICAL HARMONICS"

they are a Hilbert basis of $L^2(S^2)$

$f \in L^2(S^2) \Rightarrow f = \sum c_{\ell,m} Y_{\ell,m}$



$Y_{0,0} = 1$

$Y_{1,-1}$ $Y_{1,0}$ $Y_{1,1}$

$Y_{2,0}$

$Y_{3,0}$