LECTURE 9 MAXIMAL TORI. §5.1

ERRATA FROM LAST TIME

PROP. 4.2.3 Every connected compact Lie group with finite cntr has a compact Lie algebra

COUNTEREXAMPLE: $O_{\mathbb{R}}(2)$

$$Z(G) \subset \ker(\mathrm{Ad}) \; !$$

Def A torus T is a Lie group which is isomorphic to $(S^1)^k$, for some $k \geq 0$.

Ex 3.3 A connected Lie group G is abelian if and only if Lie G is abelian.

G abelian connected $\Rightarrow G \cong \mathbb{R}^n/\Gamma$, Γ discrete subgroup.

Thm G connected abelian Lie group,

$$G \cong (S^1)^a \times (\mathbb{R})^b.$$

Pf We just need to show that a discrete subgroup of $\mathbb{R}^n$ is generated by linearly independent vectors $\{v_i\}$

So $\Gamma = \left\{ \sum_{i=1}^{k} a_i v_i \;\middle|\; a_i \in \mathbb{Z} \right\}$

and $\mathbb{R}^n/\Gamma \cong (S^1)^k \times \mathbb{R}^{n-k}$

We show the claim by ind. on n.

$\underline{n=0}$ ✓

Γ is trivial ✓.

Assume Γ not trivial $\Rightarrow \exists\, v \in \Gamma \setminus \{0\}$, s.t. $\|v\|$ is minimal.

$(\,,\,)$ scalar product on $\mathbb{R}^n$

$p: \mathbb{R}^n \longrightarrow v^\perp$. $p(\Gamma)$ is a discrete subgroup of $v^\perp$.

$a \in p(\Gamma)$.

$$p^{-1}(a) \cap \Gamma = a + cv + \mathbb{Z}v, \quad \overset{c \in \mathbb{R}}{|a| \leq \frac{1}{2}}$$

If $p(\Gamma)$ is not discrete, $\exists\, a \in p(\Gamma)\setminus\{0\}$ s.t. $\|a\| < \frac{1}{2}\|v\|$.

$a + cv \in \Gamma$. $\|a+cv\|^2 \leq \|a\|^2 + c^2\|v\|^2 . \leq \frac{1}{4}\|v\|^2 + \frac{1}{4}\|v\|^2$

$\Rightarrow \|a+cv\| < \|v\|$. ↯ b/c v was minimal.

So $p(\Gamma)$ was discrete in $v^\perp \cong \mathbb{R}^{n-1}$

$\Rightarrow p(\Gamma) = \langle \bar{v}_1, \dots, \bar{v}_k \rangle$, $\bar{v}_1, \dots, \bar{v}_k$ ind. vectors.

$\underline{v_i \in p^{-1}(\bar{v}_i) \cap \Gamma, \Rightarrow \Gamma = \langle v, v_1, \dots, v_k \rangle.}$ $\square$

<u>Cor</u> If G is compact conn. abelian Lie group, then $\underline{G \text{ is a torus.}}$

Def Let K be a compact Lie group.
We call a maximal torus of K a subgroup $T \subset K$ which is isomorphic to $(S^1)^k$, and that it is not contained in any other torus.

Maximal torus always exist!

EXAMPLES · $SO_3(\mathbb{R}) \supset S^1_z = \{$ rotations along the z-axis $\}$

Every $v \in \mathbb{R}^3$ defines a $\overset{\text{maximal}}{\vee}$ torus by taking rotations with axis v.

All maximal tori are conjugated.

· $U_n(\mathbb{C})$. a maximal torus is given by diagonal matrices in $U_n(\mathbb{C})$

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \right\} \cong S_1^n$$

Lemma K compact Lie group. T maximal torus $\subset K$.

$$Z_K(T)^0 = T.$$

Pf $\supseteq$ is clear!

Since they are both connected, it's enough to show

$$\operatorname{Lie} Z_K(T)^0 = \operatorname{Lie} T$$

$x \in \mathrm{Lie}\, Z_h(T)$.

$$\Phi_x : \mathbb{R} \times T \longrightarrow K$$
$$(a,t) \longmapsto \underset{\substack{\cap \\ Z(T)}}{\boxed{e^{ax}}}\, t.$$

$\mathrm{Im}\,\Phi_x$ is abelian Lie group, and connected

$$e^{ax}t\, e^{a'x}t' = e^{(a+a')x}\, tt' =$$
$$e^{a'x} e^{ax} t' t = e^{a'x}t'\, e^{ax}t$$

$\overline{\mathrm{Im}\,\Phi_x}$ is an abelian compact connected.

$\Rightarrow \overline{\mathrm{Im}\,\Phi_x}$ is a torus.

and $T \subset \overline{\mathrm{Im}\,\Phi_x} \Rightarrow \overline{\mathrm{Im}\,\Phi_x} = T$.

$\Rightarrow \quad \Phi_x(a,1) \in T \quad \forall a \in \mathbb{R} \Rightarrow$

$e^{ax} \in T \ \forall a \in \mathbb{R} \Rightarrow x \in \mathrm{Lie}\, T$.

$\square$

G Lie group, N normal closed subgroup

We can take the quotient G/N.

Thm G/N is a Lie group.

We omit the proof.

H closed subgroup. G/H is a differentiable manifold.

$N \subset G$ closed normal subgroup.

$\mathrm{Lie}\, N \subset \mathrm{Lie}\, G$ is an ideal.

κ_G Killing form of G

$\kappa_G|_{\mathrm{Lie}\, N} = \kappa_N$.

Assume now N is compact.

dim = dim G $\downarrow$ dim G − dim N $\downarrow$

$$\mathrm{Lie}\, G \cong \mathrm{Lie}\, N \oplus (\mathrm{Lie}\, N)^{\perp}$$

$$p : G \to G/N \Rightarrow dp : \mathrm{Lie}\, G \xrightarrow{\sim} \mathrm{Lie}(G/N)$$

is surjective

$$dp(\mathrm{Lie}\, N) = 0 \quad \Big(\text{b/c. } dp(\dot{\gamma}(0)) = \frac{d}{dt}(\underset{\substack{\parallel \\ \text{cst.}}}{p \circ \gamma(t)})\Big|_{t=0} = 0$$

$\ker(dp) = \mathrm{Lie}\, N \Rightarrow dp$ induces an iso. between

$$(\mathrm{Lie}\, N)^{\perp} \cong \mathrm{Lie}(G/N)$$

Lem K compact Lie group, with a normal closed subgroup $N \subset K$ which is a torus, and K/N is again a torus, then K is also a torus.

Pf $\mathrm{Lie}\, K \cong \mathrm{Lie}\, N \oplus \mathrm{Lie}\, N^{\perp}$

$\mathrm{Lie}\, N^{\perp} \cong \mathrm{Lie}(K/N)$ abelian

$\Rightarrow$ Lie K is abelian $\Rightarrow$ K abelian connect copact

$\Rightarrow$ K torus $\square$.

Thm K connected cpt. Lie group.

1) $\forall x \in K$ $\exists T$ maximal torus s.t. $x \in T$.

2) If T, T' are maximal tori, then $\exists g \in K$ s.t. $gTg^{-1} \cong T'$.

Pf $T \subset K$ maximal torus in K. then gTg^{-1} is a maximal torus $\forall g \in K$. We fix a max. torus T.

We want to show

$$K = \bigcup_{g \in T} gTg^{-1}$$

We show this by ind. on K.

$\dim K = 0$, $K = \{e\}$ ✓

$\dim K > 0$, $Z = Z(K)$ center.

TZ^0 is conneted compat abelian subgroup

$\Rightarrow TZ^0$ is a torus $\Rightarrow T = TZ^0 \Rightarrow Z^0 \subset T$

$T/_{Z^0} \subset K/_{Z^0}$ is a maximal torus.

If $\dim Z^0 > 0$, $\dim(K/_{Z^0}) = \dim K - \dim Z^0 < \dim K$

$$K/_{Z^0} = \bigcup_{g \in K} g\, T g^{-1} Z^0/_{Z^0} \Rightarrow K = \bigcup_{g \in K_0} g T g^{-1}$$

$$\Downarrow$$

$x \in K \qquad x = g t g^{-1} z = g \overset{T}{\overset{\Downarrow}{\boxed{tz}}} g^{-1}.$

What happens if $Z^0 = \{e\}$?

We claim

$$\boxed{\bigcup_{g \in K} g(T \setminus Z) g^{-1} = K \setminus Z}$$

Taking closures $\bigcup_{g \in K} g T g^{-1} = K.$

Note $\bigcup g T g^{-1}$ is closed b/c it's the image of $\Phi : K \times T \to K$

$$(g, t) \mapsto g t g^{-1}.$$

If $\dim K = 1 \Rightarrow K = T.$

If $\dim K \geq 2 \Rightarrow K \setminus Z$ is connected.

So it is enough to show

$\bigcup g(T \setminus Z) g^{-1} \subset K \setminus Z$ closed and open.

· closed because

$$\bigcup g(T\setminus Z)g^{-1} = \Big(\bigcup_{g\in K} gTg^{-1}\Big)\setminus Z$$

$$T = T\setminus Z \sqcup Z \qquad \nwarrow \Phi(K\times T)$$

$$\forall g \quad gTg^{-1} = g(T\setminus Z)g^{-1} \sqcup Z$$

$$\bigcup_{g\in K} gTg^{-1} = \bigcup_{g\in K} g(T\setminus Z)g^{-1} \sqcup Z$$

· Why $\bigcup g(T\setminus Z)g^{-1}$ is open?

$t\in T\setminus Z$, $\exists U$ open nbhd $U_t \ni t$

contained in $\bigcup_{g\in G} g(T\setminus Z)g^{-1}$.

$t\notin Z$. $H := Z_K(t)^0$, $\underset{\dim H<\dim K}{T\subset H \subsetneq K}$, T maximal torus of H.

By ind. $H = \bigcup_{g\in H} gTg^{-1}$

$$\boxed{H\setminus Z = \bigcup_{h\in H} h(T\setminus Z)h^{-1}} \qquad \bigcup_{g\in K} g(H\setminus Z)g^{-1} \; \|$$

$$\bigcup_{g\in K} g(T\setminus Z)g^{-1} = \bigcup_{g\in K}\Big(\bigcup_{h\in H} gh(T\setminus Z)(gh)^{-1}\Big)$$

$$\Psi : K \times (H \setminus Z) \longrightarrow K$$

$$(g, h) \longmapsto g h g^{-1}$$

$\exists V_t \subset \mathrm{Im}(\Psi)$,

By the inverse image theorem we are done if we show that differential of Ψ is <u>surjective</u>

$$\Psi' : K \times H \longrightarrow K$$

$$(g, h) \longmapsto t^{-1} g t h g^{-1}$$

Want to show that Ψ' has surj. diff. in $(1,1)$.

$$d\Psi'_{(1,1)} : \mathrm{Lie}\,K \times \mathrm{Lie}\,H \longrightarrow \mathrm{Lie}\,K$$

$$(X, Y) \longmapsto Ad(t^{-1})X + Y - X$$

$$\begin{array}{c} \| \\ \underset{\boxed{\mathrm{Im}\,A}}{\underset{\rotatebox{90}{$\in$}}{(Ad(t^{-1}) - Id)}}\, X + \underset{\boxed{\mathrm{Lie}\,A}}{\underset{\rotatebox{90}{$\in$}}{Y}} \end{array}$$

$$H = Z_K(T)^\circ$$

$$\mathrm{Lie}\,H = \{X \in \mathrm{Lie}\,K \mid Ad(t)X = X \ \forall t\}$$

$$\mathrm{Lie}\,H \subset \ker(Ad(t^{-1}) - Id).$$

$A = \mathrm{Ad}(t^{-1}) - \mathrm{Id} : \mathrm{Lie}\, k \longrightarrow \mathrm{Lie}\, k$

$\mathrm{Lie}\, k \cong \ker A \oplus \mathrm{Im}\, A.$

A is diagonalizable!

In fact, we can find Hermitian product on $\mathrm{Lie}\, k \otimes_{\mathbb{R}} \mathbb{C}$, which is $\mathrm{Ad}(T)$-invariant.

$\Rightarrow \mathrm{Ad}(T) \subset U(\mathrm{Lie}_{\mathbb{C}}\, k)$

$\Rightarrow \mathrm{Ad}(t^{-1})$ is diagonalizable on $\mathrm{Lie}_{\mathbb{C}}\, k$.

$\Rightarrow A = \mathrm{Ad}(t^{-1}) - \mathrm{Id}$ on $\mathrm{Lie}_{\mathbb{C}}\, k$.

$$A = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \text{ on } \mathrm{Lie}_{\mathbb{C}}\, k$$

$\mathrm{Lie}_{\mathbb{C}}\, k \cong \mathrm{Im}\, A_{\mathbb{C}} \oplus \ker A_{\mathbb{C}}$

$\Rightarrow \mathrm{Lie}\, k \cong \mathrm{Im}\, A \oplus \ker A.$

$\Rightarrow d\Psi'$ is surjective. $\square$