... MAXIMAL TORI (CONTINUE)

K a compact connected Lie group

$T \subset K$ maximal torus

$$K = \bigcup_{g \in K} g T g^{-1}$$

$\Rightarrow$ $\forall h \in K$ $\exists$ max. torus T' containing h.

§ h.s

Def G top. group.

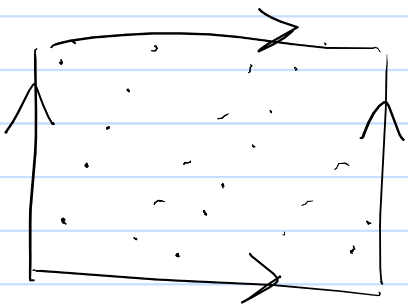
We say that G is top. cyclic if $\exists g \in G$

$\overline{\langle g \rangle} = G$ (i.e. $\langle g \rangle$ is dense in G)

We call g a top. generator.

EXAMPLES

$T = S^1 \times S^1$



Prop A compact torus T is top. cyclic.

Pf. We can assume $\mathbb{R}^k / \mathbb{Z}^k \cong T$

$a \in \mathbb{R}^n$, $\bar{a} \in T$

$a = (a_1, \dots, a_k)$

$\bar{a}$ top. generates $T \iff 1, a_1, \dots, a_k$ are alg. independent over $\mathbb{Q}$

$\bar{a}$ not a generator. $\overline{\langle a \rangle} \neq T$ is a closed subgroup

$T/\overline{\langle a \rangle}$ abelian, compact, connected $\Rightarrow T/\overline{\langle a \rangle}$ compact torus

$\cong$

$\mathbb{R}^m/\mathbb{Z}^m$ for some $m \geq 1$.

$\exists\ \overline{\Phi} : T/\overline{\langle \bar{a} \rangle} \xrightarrow{\sim} \mathbb{R}/\mathbb{Z} \cong S^1$ surjective

In other words $\exists\ \Phi : T \longrightarrow \mathbb{R}/\mathbb{Z}$ surjective

and $\Phi(\bar{a}) = 0$

We know that $\Phi : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}/\mathbb{Z}$ is of the form

$(\bar{v}_1, \dots, \bar{v}_k) \to \sum n_i \bar{v}_i$ for some $n_i \in \mathbb{Z}$

So $\Phi(\bar{a}) = 0 \iff \sum n_i \bar{a}_i = 0 \iff$

$\iff m + \sum n_i a_i = 0$ for some m

$\Rightarrow 1, a_1, \dots, a_k$ are alg. dep.

This show $\Leftarrow$

$1, a_1, \dots, a_k$ are alg. dep. Then we can find

$n_1, \dots, n_k$ as above $\rightsquigarrow$ Same proof as before $\square$

Prop K cpt. connected Lie group. S, T are max. tori
$\exists g \in K$ s.t. $gSg^{-1} = T$.

Pf. S is torus $\Rightarrow$ $s \in S$ top. generator.

$$S = \overline{\langle s \rangle}. \quad K = \bigcup_g gTg^{-1} \Rightarrow \exists g \in K \text{ s.t.}$$

$$s \in gTg^{-1} \Rightarrow \underset{\underset{S}{\parallel}}{\overline{\langle s \rangle}} \subset gTg^{-1}$$

$$\Rightarrow S \subset gTg^{-1} \Rightarrow S = gTg^{-1} \qquad \square$$

Ex. 6.4

$$\left\{ \begin{matrix} \text{maximal tori} \\ \text{in } K \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{max. abelian} \\ \text{subalgebras of Lie } K \end{matrix} \right\}$$

$$T \longmapsto \text{Lie } T$$

Def A max. abelian subalgebra is called a Cartan subalgebra

Prop Every two Cartan subalgebras in Lie K are conjugated.

Pf $\mathfrak{s}, \mathfrak{t}$ Cartan subalg. $\mathfrak{s} = \text{Lie } S$, $\mathfrak{t} = \text{Lie } T$
for some S, T max. torus.

$$\exists g \text{ s.t. } gSg^{-1} = T \Rightarrow \text{Ad}(g)\,\mathfrak{s} = \text{Ad}(g) \text{ Lie } S = \mathfrak{t}$$

$\square$

Def We call <u>rank</u> group the dimension of a max. torus.

$\operatorname{rk} K = \dim T$.

Cor If K compact, then $\exp : \operatorname{Lie} K \to K$ is surjective

Pf. $g \in K$. $\exists T \ni g$ maximal torus

$\exp : \operatorname{Lie} T \to T$ is surjective

$g \in \exp(\operatorname{Lie} T) \subset \exp(\operatorname{Lie} G)$. $\square$

Example $\exp : \mathfrak{gl}_2(\mathbb{R}) \to GL_2(\mathbb{R})$

CLASSIFICATION IN Rk 1 §5.2.

Thm Let K be compact connected Lie group with max. torus T. If $\dim T \cong S^1$.

Then K is isomorphic to $SO_3(\mathbb{R})$, $SU_2(\mathbb{C})$ or S^1.

Pf. $\dim K \geq 1$.

If $\dim K = 1 \Rightarrow K = T$ b/c K, T connected.

$\Rightarrow K \cong S^1$.

$\dim K > 1$.

$\mathfrak{g} = \operatorname{Lie} K \otimes_{\mathbb{R}} \mathbb{C}$, $c : \mathfrak{g} \to \mathfrak{g}$

$X \otimes z \mapsto X \otimes \bar{z}$

$\mathrm{Lie}\, K = \mathfrak{g}^c = \{X \in \mathfrak{g} \mid c(X) = X\}$.

$T \cong S^1$. (we fix an isomorphism χ)

T acts via Ad on $\mathfrak{g}$.

so we can decompose $\mathfrak{g}$ into irred. T reps.

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \qquad \mathfrak{g}_i = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t) X = \chi(t)^i X\}$$

•) $[\mathfrak{g}_n, \mathfrak{g}_m] \subset \mathfrak{g}_{n+m}$

$$\mathrm{Ad}\, t([X,Y]) = [\mathrm{Ad}(t) X, \mathrm{Ad}(t) Y] =$$
$$= [t^n X, t^m Y] = t^{n+m} [X,Y]$$

•) $c(\mathfrak{g}_m) = \mathfrak{g}_{-m}$

$\mathrm{Ad}\, t : \mathrm{Lie}\, K \to \mathrm{Lie}\, K$ so it commutes with c

$X \in \mathfrak{g}_m$

$$\mathrm{Ad}\, t(c(X)) = c\, \mathrm{Ad}(t)(X) = c(\chi(t)^m X) =$$
$$\chi(t)^{-m} c(X)$$

$\Rightarrow c(X) \in \mathfrak{g}_{-m}$.

$\cdot)$ $\mathfrak{g}_0 = \mathrm{Lie}\,T \otimes_{\mathbb{R}} \mathbb{C}$.

$$T = Z_K(T)^\circ \Rightarrow \mathrm{Lie}\,T = \mathrm{Lie}\,Z_K(T)$$
$$\mathrm{Lie}\,Z_K(T) = \{X \in \mathfrak{g} \mid (\mathrm{Ad}\,t)X = X \ \forall t \in T\}$$

$$\mathrm{Lie}\,T = \mathfrak{g}_0 \cap \mathrm{Lie}\,K = \mathfrak{g}_0^c$$

$$c : \mathfrak{g}_0 \to \mathfrak{g}_0 \quad \Rightarrow \quad \mathfrak{g}_0 = \mathfrak{g}_0^c \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Lie}\,T \otimes_{\mathbb{R}} \mathbb{C}$$

$$\dim_{\mathbb{C}} \mathfrak{g}_0 = \underline{1} \Rightarrow \dim_{\mathbb{C}} \mathfrak{g} = \dim \mathfrak{h} > 1$$

$\exists\, m \neq 0$ s.t. $\mathfrak{g}_m \neq 0$ $\quad \cancel{\neq 0}$

Choose $m > 0$ s.t. $\mathfrak{g}_m \neq 0$. Let $X \in \mathfrak{g}_m$.

$0 \neq c(X) \in \mathfrak{g}_{-m}$.

$[X, c(X)] \in \mathfrak{g}_0$.

<u>Goal</u> Show $\mathfrak{g} = \langle X, c(X), [X, c(X)] \rangle$

$\cdot$ $[X, c(X)] \neq 0$.

Assume is 0. Then $\mathfrak{h} = \langle X, c(X) \rangle$ is an abelian Lie algebra.

$c : \mathfrak{h} \to \mathfrak{h}$, $\mathfrak{h}^c$ is an abelian subalgebra of $\dim 2_{/\mathbb{R}}$

$\mathfrak{h}^c \subset \mathrm{Lie}\,K$

$\mathfrak{h}^c = \langle X + c(X), iX - ic(X) \rangle$ $\quad ⚡$

$\Rightarrow [X, c(X)] \neq 0$.

define $V = \mathbb{C}c(X) \oplus \bigoplus_{m \geq 0} \mathfrak{g}_m$

- V is stable under $\underbrace{\mathrm{ad}X}$ and $\mathrm{ad}\, c(X)$

$\mathrm{ad}X : \mathfrak{g}_h \to \mathfrak{g}_{m+h}$ $\forall h$ ✓

$\mathrm{ad}\, c(X) : \mathfrak{g}_h \to \mathfrak{g}_{h-m}$

we need to check what is $\mathrm{ad}\, c(X)|_{\mathfrak{g}_0}$.

$\mathfrak{g}_0 = \mathrm{Lie}\, T \otimes_{\mathbb{R}} \mathbb{C}$.

$Y \in \mathrm{Lie}\, T$ $\quad e^{tY} \in T \ \forall t \in \mathbb{R}$

How $\mathrm{ad}Y$ acts on $\mathfrak{g}_h$? $Z \in \mathfrak{g}_k$.

$$\mathrm{ad}Y(Z) = \frac{d}{dt}\left(\mathrm{Ad}(e^{Yt})Z\right)\Big|_{t=0} =$$

$$= \frac{d}{dt} \chi(e^{Yt})^{k} Z\Big|_{t=0} =$$

$$= k\, \underbrace{\chi(e^{0})}_{=1}^{k-1} d\chi(Y)\, Z =$$

$$= k\, d\chi(Y)\, Z$$

$\dim \mathfrak{g}_0 = 1$, and it is generated $[X, c(X)]$

$Y \in \mathfrak{g}_0$, $\operatorname{ad} c(X)(Y) = -\operatorname{ad} Y\, c(X) = m\, d\cancel{X}(Y)\, c(X)$
$$\underbrace{}_{\subseteq V}$$

$\Rightarrow$ V is stable under $[\operatorname{ad} X, \operatorname{ad} c(X)] = \operatorname{ad}([X, c(X)])$

$\operatorname{ad}([X, c(X)])$ has zero trace on V,
and the same is true $\forall Y \in \mathfrak{g}_0$. $\operatorname{tr} Y|_V = 0$

but $V = \mathbb{C}\, c(X) \oplus \bigoplus_{n \geq 0} \mathfrak{g}_n$

$$0 = \operatorname{tr} Y|_V = -m\, d\cancel{X}(Y) + \sum_{n \geq 0} n\, d\cancel{X}(Y) \dim_{\mathbb{C}} \mathfrak{g}_n$$

$\dim_{\mathbb{C}} \mathfrak{g}_m \geq 1 \Rightarrow \dim \mathfrak{g}_m = 1$, $\dim \mathfrak{g}_n = 0\ \forall n > m$.

$\Rightarrow V = \mathfrak{g}$. $\Rightarrow \dim_{\mathbb{C}} \mathfrak{g} = 3 \Rightarrow \dim \mathfrak{h} = 3$.

We also know that $Z(\operatorname{Lie} K) = \{0\}$
o/w $Z \in Z(\operatorname{Lie} K)$, $X \in \mathbb{C} Z$,
$\langle X, Z \rangle$ is an abelian Lie algebra. $\lightning$

$\operatorname{Ad} : K \to GL(\operatorname{Lie} K)$

$d\operatorname{Ad} = \operatorname{ad} : \operatorname{Lie} K \to \mathfrak{gl}(\operatorname{Lie} K) \Rightarrow d\operatorname{Ad}$ is injective.

K is compact, we can an K-inv. scalar product on Lie K

$$\Rightarrow \quad \varphi : K \to SO(\mathrm{Lie}\,K) \cong SO_3(\mathbb{R}), \quad d\varphi \text{ injective}$$

$\dim K = 3 = \dim SO_3(\mathbb{R})$.

$\Rightarrow d\varphi$ is isomorphism.

$\mathrm{Lie}\,K \cong \mathrm{Lie}\,SO_3(\mathbb{R})$. $\qquad$ φ surj

$\ker\varphi$ is a discrete group, so it is finite.

$$\begin{tikzcd} K \arrow[r] & SO_3(\mathbb{R}) \\ \mathrm{Lie}\,K \arrow[u, "\exp"] \arrow[r, "\sim"] & \mathrm{Lie}\,SO_3(\mathbb{R}) \arrow[u, "\exp"'] \end{tikzcd}$$

exp surjective

$\Rightarrow \varphi$ surjective

<u>Lem</u> K compact conneted Lie group.

$\varphi : K \to SO_3(\mathbb{R})$ surjective.

$\ker\varphi$ is finite. $\Rightarrow K \cong SO_3(\mathbb{R})$ or $K \cong SU_2(\mathbb{C})$.

<u>Pf</u> $K \not\cong SO_3(\mathbb{R})$. $\exists\, g \in \ker K$, $g \neq \mathrm{id}$.

Using Peter-Weyl thm as in Exercise 5.4 we can find an irreducible complex rep. V of H s.t.

g acts not trivially.

$\mathrm{Lie}\, H \simeq \mathrm{Lie}\, SO_3(\mathbb{R})$

$\mathrm{Lie}\, H \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{sl}_2(\mathbb{C})$.

We know the irr. rep. of $\mathfrak{sl}_2(\mathbb{C})$

$L(m)$, $m \geq 0$, $L(m)$ has dim $m+1$.

We also know that $L(m)$ comes from a rep. of $SO_3(\mathbb{R})$ if and only if m is even.

So $V \simeq L(m)$ as a $\mathfrak{sl}_2(\mathbb{C})$-rep.

and m is not even. (If m is even we would have such a diagram

$$\begin{tikzcd} H \arrow[r] & SO_3(\mathbb{R}) \arrow[r] & GL(L(m)) \\ \mathrm{Lie}\, H \arrow[u] \arrow[r, "\sim"] & \mathrm{Lie}\, SO_3(\mathbb{R}) \arrow[u] \arrow[r] & \mathfrak{gl}(L(m)) \arrow[u] \end{tikzcd}$$

but then g would act trivially.

So m is odd $\Rightarrow L(m)$ is even dimensional.

$$\begin{tikzcd}
K \arrow[r, "\rho"] & GL(L(m)) & SU_2(\mathbb{C}) \arrow[l, "\rho'"'] \\
\operatorname{Lie} K \arrow[u] \arrow[r, "d\rho"] \arrow[rr, bend right, "\sim"] & \mathfrak{gl}(L(m)) \arrow[u] & \operatorname{Lie} \mathfrak{su}_2(\mathbb{C}) \arrow[u] \arrow[l, "d\rho'"']
\end{tikzcd}$$

I claim that ρ' is injective.

We know that $\operatorname{Lie} \mathfrak{su}_2(\mathbb{C})$ has no ideals.
(EXERCISE 4.4)

$\Rightarrow d\rho'$ injective

$\Rightarrow \underset{\substack{\parallel \\ H}}{\ker(\rho')}$ is finite

H finite normal subgroup of $SU_2(\mathbb{C})$

$\Rightarrow H$ is in the center.

$$SU_2(\mathbb{C}) \longrightarrow S_H$$
$$g \longmapsto (h \mapsto g h g^{-1})$$

$\Rightarrow Z(SU_2(\mathbb{C})) = \{\pm \mathrm{id}\}$

But if $\ker \rho = \{\pm \mathrm{id}\}$ then ρ factorizes through $SO_3(\mathbb{R})$.

$$\begin{tikzcd}
K \arrow[r, "\rho"] & GL(L(m)) & SU_2(\mathbb{C}) \arrow[l, "\rho'"'] \\
\mathrm{Lie}\,K \arrow[u] \arrow[r, "d\rho"] \arrow[rr, bend right, "\sim"] & \mathfrak{gl}(L(m)) \arrow[u] & \mathrm{Lie}\,SU_2(\mathbb{C}) \arrow[u] \arrow[l, "d\rho'"']
\end{tikzcd}$$

$$\rho(K) = \rho(\exp \mathrm{Lie}\,K) = \exp(d\rho(\mathrm{Lie}\,K)) = \rho'(SU_2(\mathbb{C}))$$

$$\rho : K \to \rho'(SU_2(\mathbb{C})) \cong SU_2(\mathbb{C}).$$

so ρ is a group hom.

Why is it a bijection?

O/w we can find $g \in \ker \rho$.

and find irr. rep. of K on which g acts non-trivially.

$\Updownarrow$

V

But V is an irr. rep. of $\mathfrak{sl}_2(\mathbb{C})$,

so it is also a rep. of $SU_2(\mathbb{C})$.

But then this means V factors though ρ.

So g acts trivially on V b/c $g \in \ker \rho$ $\square$