LECTURE 12:

ROOT SYSTEMS AND CLASSIFICATION

Prop K compact connected Lie group
$\cup$
T max torus

$R := R(K,T) \subset X(T)$

$W := W(K,T) = N_K(T)/T$

1) $\forall \alpha \in R$ $\quad \dim(\mathrm{Lie}_{\mathbb{C}} K)_\alpha = 1$,
and if $\beta \in R$ is a multiple of α then $\beta = \alpha$ or $-\alpha$.

2) $\exists!\ s_\alpha \in W$ which acts on X as a reflection
with $s_\alpha(\alpha) = -\alpha$

3) $\exists!\ \alpha^\vee : X \to \mathbb{Z}$ s.t. $s_\alpha(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha^\vee \rangle}_{\alpha^\vee(\lambda).} \alpha \quad \forall \lambda \in X$

Pf. $\alpha : T \to S^1$. $\quad S = (\ker \alpha)^0 \subset T$

S compact, connected, abelian $\Rightarrow$ S torus.

$\dim S = \dim(\ker \alpha) = \dim T - 1$

$$\begin{array}{ccc} T & \hookrightarrow & Z_K(S) \\ \downarrow & & \downarrow \\ T/S & \longrightarrow & Z_K(S)/S \end{array}$$

Last time: surjective morphisms send max. tori to max. tori.

T max. torus in $K \Rightarrow T$ max. torus in $Z_K(S)$

$\Rightarrow T/S$ max. torus $Z_K(S)/S$

$\dim(T/S) = 1 \Rightarrow T/S \cong S^1$

$Z_K(S)/S$ is a compact group of rk 1.

$$\mathrm{Lie}_{\mathbb{C}} Z_K(S) \supset \mathrm{Lie}_{\mathbb{C}} Z_K(\ker \alpha)$$
$$= \{X \in \mathrm{Lie}_{\mathbb{C}} K \mid (\mathrm{Ad}\, h) X = X \ \forall h \in \ker \alpha\}$$
$$\cup$$
$$(\mathrm{Lie}_{\mathbb{C}} K)_\alpha$$
$$\in$$

because for X, $h \in T$ acts as $(\mathrm{Ad}\, h)(X) = \alpha(h) X$

$\mathrm{Lie}_{\mathbb{C}} Z_K(S) \supsetneq \mathrm{Lie}\, T$

$\Rightarrow T/S \neq Z_K(S)/S \Rightarrow Z_K(S)/S \cong SO(3)$ or $SU(2)$.

In both cases

$$R(Z_h(S)/_S, T/_S) = \{\alpha, -\alpha\}$$

$$\mathrm{Lie}_{\mathbb{C}} Z_h(S) = \mathrm{Lie}_{\mathbb{C}} Z_h(S)_0 \oplus \mathrm{Lie}_{\mathbb{C}} Z_h(S)_\alpha \oplus \mathrm{Lie}_{\mathbb{C}} Z_h(S)_{-\alpha}$$

$$\Rightarrow R(Z_h(S), T) = \{\alpha, -\alpha\}$$

$$\underbrace{\mathrm{Lie}_{\mathbb{C}}(Z_h(S)/_S)_\alpha}_{\text{has dim } 1} \cong \mathrm{Lie}_{\mathbb{C}} Z_h(S)_\alpha \quad \Rightarrow \quad \underline{\mathrm{Lie}_{\mathbb{C}} Z_h(S)_\alpha = 1}$$

$$\mathrm{Lie}_{\mathbb{C}} Z_h(S) = \{X \in \mathrm{Lie}_{\mathbb{C}} h \mid Ad(h)X = X \ \forall h \in S\}$$

$$= (\mathrm{Lie}_{\mathbb{C}} h)^{Ad(S)} = (\mathrm{Lie}_{\mathbb{C}} k)^{ad(\mathrm{Lie}\, S)}$$

$$\mathrm{Lie}_{\mathbb{C}} k = \bigoplus_{\lambda \in X} (\mathrm{Lie}_{\mathbb{C}} h)_\lambda$$

$X \in (\mathrm{Lie}_{\mathbb{C}} h)_\lambda$, $Y \in \mathrm{Lie}\, S = \mathrm{Lie}(\ker \alpha) = \ker d\alpha$

$$ad(Y)X = \frac{d}{dt} Ad(e^{tY}) X \Big|_{t=0} =$$

$$= \frac{d}{dt} \lambda(e^{tY}) X = d\lambda(Y) X$$

$$(\mathrm{Lie}_{\mathbb{C}} k)^{\mathrm{ad}(\mathrm{Lie}\, S)} = \bigoplus_{\ker d\beta \supset \ker d\alpha} (\mathrm{Lie}_{\mathbb{C}} h)_\beta$$

↖ i.e. the ones which are multiple of α

$$\mathrm{Lie}_{\mathbb{C}} Z_h(S) = \mathrm{Lie}_{\mathbb{C}} Z_h(S)_0 \oplus \mathrm{Lie}_{\mathbb{C}} Z_h(S)_\alpha \oplus \mathrm{Lie}_{\mathbb{C}} Z_h(S)_{-\alpha}$$

$$\mathrm{Lie}_{\mathbb{C}} Z_h(S) \subseteq \bigoplus_{\beta \propto \alpha} \mathrm{Lie}_{\mathbb{C}}(h)_\beta$$

$$\mathrm{Lie}_{\mathbb{C}} Z_h(S)_\alpha \subseteq (\mathrm{Lie}_{\mathbb{C}} h)_\alpha$$

$\Rightarrow$ If β is a multiple of α, $\beta \neq 0$, $\beta = \alpha$ or $\beta = -\alpha$

$$\mathrm{Lie}_{\mathbb{C}} h_\alpha = \mathrm{Lie}_{\mathbb{C}} Z_h(S)_\alpha$$

$$\dim \mathrm{Lie}_{\mathbb{C}} h_\alpha = 1$$

Rmk $\mathrm{Lie}\, Z_h(\ker \alpha) = \mathrm{Lie}\, Z_h(S)$

$Z_h(S)$ is connected

$\Rightarrow Z_h(\ker \alpha) = Z_h(S)$

$$\begin{tikzcd}
s_\alpha \in W\left(Z_h(S)/S, T/S\right) \cong W\left(Z_h(S), T\right) = N_{Z_h(S)}(T)/T \arrow[d, hook] \\
W(h, T) = N_h(T)/T
\end{tikzcd}$$

We have the following s.e.s.

$$\mathbb{Z} \cong X(T/S) \hookrightarrow \underset{\mathbb{Z}^n}{\overset{}{X(T)}} \twoheadrightarrow \underset{\mathbb{Z}^{n-1}}{X(S)}$$

$$X(T) \cong \mathbb{Z}^n, \qquad X(S) \cong \mathbb{Z}^{n-1}$$

s_α acts $X(T/S)$ as mult. by -1.

$s_\alpha \in N_{Z_h(S)}T/T \rightsquigarrow \exists\, \dot{s}_\alpha$ representative in $Z_h(S)$

so $\dot{s}_\alpha$ acts trivially on $X(S)$.

$\lambda : S \to S^1$ $\qquad s_\alpha \cdot \lambda(x) = \lambda(\dot{s}_\alpha x \dot{s}_\alpha^{-1}) = \lambda(x)$

There exists a basis of $X(T)$ such that in this basis

s_α is represented by $\begin{pmatrix} -1 & * & * & \times & \cdots \\ & 1 & & & 0 \\ & & 1 & & \\ 0 & & & 1 & \\ & & & & 1 \end{pmatrix}$

so s_α^2 is unipotent, and in a finite group

unipotent $\Rightarrow$ identity.

So $s_\alpha^2 = \mathrm{Id}$, , $s_\alpha^2 = \mathrm{Id}$, $s_\alpha = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$

We see that s_α is a reflection, and

$s_\alpha(\alpha) = -\alpha$.

It is unique: if t is another such reflection with $t(\alpha) = \alpha$

then $ts(\alpha) = \alpha$

and both s and t are identity on $\mathcal{X}(T)/\langle\alpha\rangle$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$
$\mathcal{X}(T/S)$

$\Rightarrow$ st is unipotent $\Rightarrow$ st identity $\Rightarrow$ $s = t$.

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3) $\alpha : T \to S^1$

$\ker\alpha \hookrightarrow T \twoheadrightarrow S^1$

$\boxed{Z_n(S) = Z_n(\ker\alpha)}$

$s_\alpha\ \mathcal{X}(S^1) \hookrightarrow \mathcal{X}(T) \twoheadrightarrow \mathcal{X}(\ker\alpha)$

$\mathbb{Z}\alpha$

$\Phi : \mathcal{X}(T) \to \mathcal{X}(T)$ $\quad\quad$ $\mathrm{Im}\,\Phi \subset \mathbb{Z}\alpha$

$\lambda \longmapsto (\lambda - s(\lambda))$.

$\alpha^\vee(\lambda) := \dfrac{\lambda - s(\lambda)}{\alpha}$, $\alpha^\vee : \mathcal{X}(T) \longrightarrow \mathbb{Z}$

$\langle \alpha^\vee, \alpha\rangle = \dfrac{\alpha - s_\alpha(\alpha)}{\alpha} = 2$, $\alpha^\vee \circ s(\lambda) = \dfrac{s(\lambda) - s^2(\lambda)}{\alpha} = -\alpha^\vee$

We check that $\operatorname{Im} \Phi \subset \mathbb{Z}\alpha$

$$s \curvearrowright X(S^1) \hookrightarrow X(T) \twoheadrightarrow X(\ker \alpha)$$

$\mathbb{Z}\alpha$

s acts as -1, How does it act on $X(\ker \alpha)$?

$\lambda : \ker \alpha \to S^1$. $\quad Z_K(\ker \alpha) = Z_K(S) \ni \dot{s}$

$$s \cdot \lambda(x) = \lambda(\dot{s} x \dot{s}^{-1}) = \lambda(x)$$

s fixes $X(\ker \alpha)$.

$\Rightarrow s^2 = \mathrm{Id} \Rightarrow \operatorname{Im} \Phi = (\lambda \mapsto \lambda - s(\lambda)) = \mathbb{Z}\alpha$.

$$s(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha^\vee \rangle}_{= \frac{\lambda - s(\lambda)}{\alpha}} \alpha \iff \langle \lambda, \alpha^\vee \rangle = \frac{\lambda - s(\lambda)}{\alpha}$$

G compact connected Lie group w/ max. torus T

$W(G,T) \circlearrowleft \operatorname{Lie} T$

$\parallel$

$N_G(T)/T \ni w \rightsquigarrow \dot{w} \in N_G(T)$

$$w : T \longrightarrow T \quad \rightsquigarrow \quad \operatorname{Lie} T \rightarrow \operatorname{Lie} T$$

$$t \longmapsto \dot{w} t \dot{w}^{-1} \qquad t \longmapsto \operatorname{Ad}(\dot{w}) t = \dot{w} t \dot{w}^{-1}$$

Let $s_\alpha \in W$ reflection corresponding to a root α

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \forall \lambda \in X(T).$$

How does s_α act on $\operatorname{Lie} T$?

s_α is a reflection on $\operatorname{Lie} T$ which fixes $\ker(d\alpha)$

$$\alpha : T \rightarrow S^1$$

$$d\alpha : \operatorname{Lie} T \rightarrow \operatorname{Lie} S^1 \cong \mathbb{R}$$

<u>Why?</u> $X \in \ker d\alpha$, $\dot{s}$ representative for s in $N_G(T)$.

Want to show that $\dot{s} X \dot{s}^{-1} = X$.

$\Updownarrow$

$$\dot{s} e^{tX} \dot{s}^{-1} = e^{tX} \quad \forall t \in \mathbb{R}$$

$$\Leftrightarrow \forall \lambda \in X(T), \quad \lambda(\dot{s} e^{tX} \dot{s}^{-1}) = \lambda(e^{tX}) \; \forall \lambda \in X(T)$$

$$\Leftrightarrow \forall \lambda \in X(T), \quad \lambda(\dot{s}\, e^{tX} \dot{s}^{-1}) = \lambda(e^{tX}) \ \forall \lambda \in X(T)$$

$$\Leftrightarrow \quad s(\lambda)(e^{tX}) = \lambda(e^{tX}) \quad \forall \lambda \in X(T), \forall t \in \mathbb{R}$$

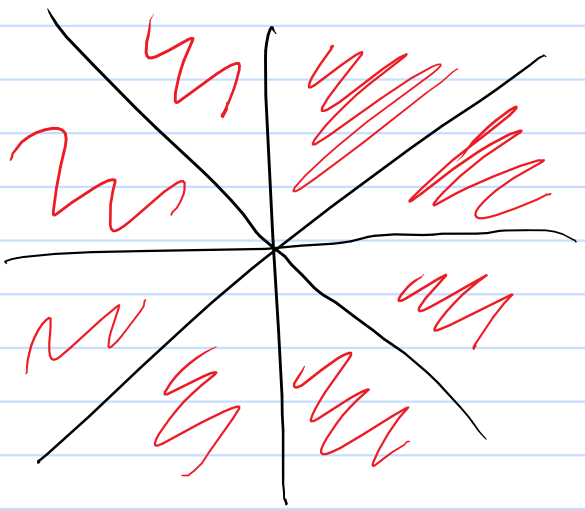
<u>Recall that</u> $\lambda - s(\lambda) = \langle \lambda, \alpha^\vee \rangle \alpha$

$$\Leftrightarrow \quad 1 = \lambda(e^{tX}) \big/ s(\lambda)(e^{tX}) = \alpha(\underbrace{e^{tX}}_{\in \ker \alpha})^{\langle \lambda, \alpha^\vee \rangle} = 1$$

s fixed $\ker(d\alpha)$, $s^2 = \mathrm{Id}$, $s \neq \mathrm{id}$

$\Rightarrow s_\alpha$ acts as a reflection on Lie T.

$$\mathrm{Lie}\, T \setminus \bigcup_{\alpha \in R} \ker d\alpha$$



We call the connected components of this set "alcoves".

Let $\mathcal{A}$ be the set of alcoves.

Prop H, T, S_α as before.

1) W generated by reflections
2) S_α, for $\alpha \in R$, are the only elements in W acting as reflections on Lie T.
3) W acts freely and transitively on the set of alcoves $\mathcal{A}$

Pf. $W' = \langle S_\alpha \mid \alpha \in R \rangle$. A priori $W' \subset W$

We want to show $W' = W$.

1st claim W acts freely on $\mathcal{A}$.

Assume $w \in W$ fixes an alcove $A \in \mathcal{A}$.

Then, there exists also $p \in A$ which is fixed by w.

$\Rightarrow w \in \mathrm{Stab}(p)$.

But if $p \in \mathrm{Lie}\, T$, $p \notin \ker d\alpha$ $\forall \alpha \in R$.

$\Rightarrow \mathrm{Lie}\, Z_H(p) = \mathrm{Lie}\, T$

$\mathrm{Lie}\, Z_H(p)_\alpha = 0 \ \forall \alpha \Rightarrow \mathrm{Lie}\, Z_H(p) = \mathrm{Lie}\, T$

We know that $Z_h(p)$ is connected.

$T \subset Z_h(p)$.

$\mathrm{Lie}\, T = \mathrm{Lie}\, Z_h(p) \Rightarrow T = Z_h(p)$.

$w \in W \rightsquigarrow \dot{w} \in N_h(T)$, $\dot{w} p \dot{w}^{-1} = p$

$\Rightarrow \dot{w} \in Z_h(p) = T \Rightarrow w = \mathrm{id}$

2ND CLAIM The action of W' is transitive.

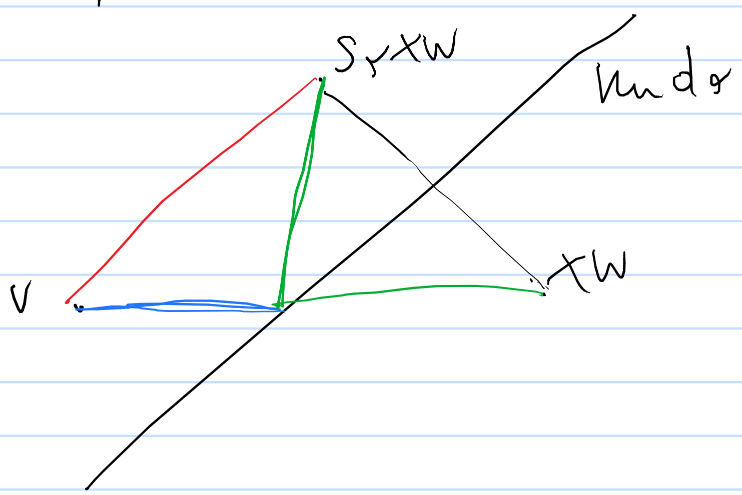
Fix an W-invariant scalar product on $\mathrm{Lie}\, T$

Fix $v, w \in \mathrm{Lie}\, T$. Choose $x \in W'$ s.t.

$\|v - xw\|$ is minimal.

If v and xw are separated by some hyperplane

$\ker(d\varphi)$, then $\|v - s_\varphi x w\| < \|v - xw\|$



So v, xw are not separated by hyperplanes

$\Rightarrow$ v, xw are in the same alcove

$\Rightarrow$ action of W' is transitive.

$W' \subset W \Rightarrow W$ acts transitively.

But W also acts freely. Fix $v \in \mathrm{Lie}\, T$

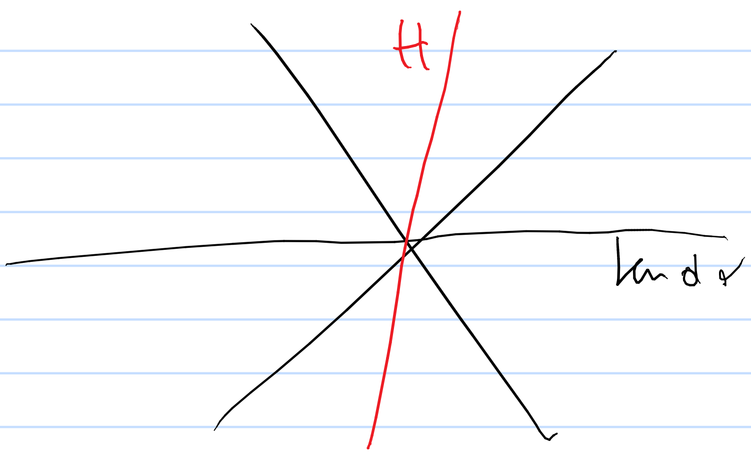
$w \in W \setminus W'$. $\exists w' \in W'$ such that

v and $w'wv$ are in the same alcove

$\Rightarrow w'w = id \Rightarrow w = (w')^{-1} \in W'$. $\lightning$

$\Rightarrow W = W'$.

2) If s is a reflection, which is different from any s_α, then also its fixed hyperplane H is different from any $\ker d\alpha$.



H intersects some alcove $A \in \mathcal{A}$.

$\Rightarrow sA = A \Rightarrow s = \mathrm{id}$ because action is free $\square$.

CLASSIFICATION

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{simple connected compact} \\ \text{Lie groups w/ trivial center}\end{array}\right\} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \left\{\begin{array}{l}\text{simple compact real} \\ \text{Lie algebras}\end{array}\right\} \arrow[dl, leftrightarrow] \\
& \left\{\begin{array}{l}\text{indecomposable root} \\ \text{system}\end{array}\right\} &
\end{tikzcd}$$

compact Lie groups w/ trivial center	simple compact Lie algebras	indecomposable root systems
$SU_{n+1}(\mathbb{C}) / Z(SU_{n+1}(\mathbb{C}))$	su_n	A_n, $n \geq 1$ n = rank
$SO_{2n+1}(\mathbb{R})$	so_{2n+1}	B_n, $n \geq 2$
$Sp(2n, \mathbb{C}) \cap SU(2n) / \pm id$	sp_{2n}	C_n, $n \geq 3$.
$SO_{2n}(\mathbb{R}) / \{\pm id\}$	so_{2n}	D_n, $n \geq 4$

EXCEPTIONAL CASES

$\dim G_{E_6} = 78$ $\longleftarrow$	E_6
133	E_7
248	E_8
52	F_4
14	G_2

$$\left\{ \begin{array}{l} \text{connected} \\ \text{compact} \\ \text{Lie groups} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{root datum:} \\ R \hookrightarrow X(T) \end{array} \right\}$$

$SU_2(\mathbb{C})$, $SO_3(\mathbb{R})$ have the same root system A_1
but it sits differently in $X(T)$

The groups of rank 3 and root system A_3

$SU_4(\mathbb{C})$, $PSU_4(\mathbb{C})$, $SU_4(\mathbb{C})/\{\pm id\}$

The groups of rank 4 and root system D_4

$Spin_8(\mathbb{R})$, $SO_8(\mathbb{R})$, $SO_8(\mathbb{R})/\{\pm id\}$