Notes on the Maschke theorem and the Peter–Weyl theorem for Lie groups

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These are notes written for the course "Lie Groups" held at the University of Freiburg in SS 23. These are mostly bases on [1, 2, 5, 6]

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1 Haar measure

Remark 1.1 (Motivation from the representation theory of finite groups)**.** If G is a finite group, there is the averaging operator

$$\begin{aligned} \Sigma : \{f : G \to \mathbb{R}\} &\to \mathbb{R} \\ f &\mapsto \frac{1}{|G|}\sum_{g\in G} f(g) \end{aligned} \tag{1}$$

Let V be a finite dimensional representation of G. If $\langle -,-\rangle$ is scalar product on V, we obtain a left G-invariant scalar product on V by setting

$$b(v,w) := \Sigma(\langle gv, gw\rangle) = \frac{1}{|G|}\sum_{g\in G}\langle gv, gw\rangle.$$

Using b one can define the orthogonal of a subrepresentation and thus prove Maschke's theorem. The goal of this section is to define an analogous averaging operator for compact Lie groups.

Let X be a topological space. Let $f : X \to \mathbb{R}$ a continuous function. The *support* of f is $\mathrm{supp}(f) = \overline{\{x \in X | f(x) \neq 0\}}$. We denote by $\mathcal{C}_!(X, \mathbb{R})$ the vector space of continuous real functions on X with compact support.

Definition 1.2. A *Radon measure* on X is a linear function $\Lambda : \mathcal{C}_!(X, \mathbb{R}) \to \mathbb{R}$ such that if $f(x) \geq 0$ for all $x \in X$ we have $\Lambda(f) \geq 0$.

Let now G be a topological group. For $g \in G$ we denote by $L_g : G \to G$ the left multiplication by g.

Definition 1.3. A *Haar measure* H on G is a non-trivial left G-invariant Radon measure A. In formulas, this means that for any $f \in \mathcal{C}_!(X, \mathbb{R})$ and $g \in G$ we have $H(f) = H(f \circ L_g)$.

Example 1.4.

- If G is a finite group, the averaging operator Σ from (1) is a Haar measure.
- On the group $(\mathbb{R}, +)$, the Lebesgue measure is a Haar measure. In fact, for any $f \in \mathcal{C}_!(X, \mathbb{R})$ and $y \in \mathbb{R}$ we have $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x + y)dx$.
- On the group S^1 a Haar measure is given by $f \mapsto \int_0^{2\pi} f(e^{i\theta})d\theta$.
- On the group $(\mathbb{R}_{>0}, \cdot)$, a Haar measure is given by $f \mapsto \int_0^\infty \frac{f(x)}{x}dx$. In fact, for $y > 0$ we have
$$\int_0^\infty \frac{f(xy)}{x}dx = \int_0^\infty \frac{f(z)y}{z}\frac{dz}{y} = \int_0^\infty \frac{f(z)}{z}dz$$

Theorem 1.5 (General existence and uniqueness of the Haar measure)**.** *Let G be a locally compact Hausdorff topological group. Then, there exists a Haar measure on G. Moreover, the Haar measure is unique up to multiplication by a positive real number.*

A general proof for topological groups see [1, Satz 4.3.8]. Here we prove instead a weaker version, which is enough for our purposes.

Definition 1.6. We call a *continuous positive density* on G a linear function $F : \mathcal{C}_!(G, \mathbb{R}) \mapsto \mathbb{R}$ if for any chart $\phi : U \to G$ there exists $q_U \in \mathcal{C}(U, \mathbb{R}_{>0})$ such that for any $h \in \mathcal{C}_!(\phi(U), \mathbb{R})$ we have $F(h) = \int_U q_U(h \circ \phi)$, where $\int_U$ denotes the Lebesgue measure on U.

Let $\psi : V \to G$ be another chart and let $h \in \mathcal{C}_!(\phi(U) \cap \psi(V))$. Then by the change of coordinates under integral (see [3, Satz 8.1.9]) we have

$$F(h) = \int_U q_U \cdot (h \circ \phi) = \int_V (q_U \circ \phi^{-1} \circ \psi) \cdot (h \circ \psi)|\det d(\phi^{-1} \circ \psi)| \tag{2}$$

from which it follows that $q_V = (q_U \circ \phi^{-1} \circ \psi)|\det d(\phi^{-1} \circ \psi)|$ on the intersection.

Remark 1.7. If $g \in G$ and F is a positive continuous density and so is $F_g(f) := F(f \circ R_g)$. In fact, if ψ is a chart, also $R_g \circ \psi$ is a chart, and we have for $h \in \mathcal{C}_!(U, \mathbb{R})$ that

$$F_g(h \circ (R_g \circ \psi)^{-1}) = F(h \circ \psi^{-1} \circ (R_g)^{-1} \circ R_g) = F(h \circ \psi^{-1}) = \int_U q_\psi h.$$

Similarly, also $F'(f) = F(f \circ \mathrm{inv})$ is a positive continuous density because if ψ is a chart also $\mathrm{inv} \circ \psi$ is a chart.

Theorem 1.8 (Existence and uniqueness of continuous Haar density on Lie groups)**.** *Let G be a Lie group. Then, there exists a continuous positive density on G which is a Haar measure on G. Moreover, such a continuous density is unique up to multiplication by a positive real number.*

Proof. We first prove the uniqueness. Let μ and ν be two Haar measures which are continuous positive densities.

Fix a chart $\phi : U \to G$. Then for any $h \in \mathcal{C}_!(\phi(U), \mathbb{R})$ we have $\mu(h) = \int_U q_\mu(h \circ \phi)$ and $\nu(h) = \int_U q_\nu(h \circ \phi)$ for some $q_\mu, q_\nu \in \mathcal{C}(U, \mathbb{R}_{>0})$. Let $r = \frac{q_\nu}{q_\mu} \in \mathcal{C}(U, \mathbb{R}_{>0})$.

For $p, q \in U$, we can find $g \in G$ such that $g(\phi(p)) = \phi(q)$ and $V_p, V_q \subset U$ neighborhoods of p and q such that $g(\phi(V_p)) = \phi(V_q)$. Let $r_g = r \circ \phi^{-1} \circ L_{g^{-1}} \circ \phi \in \mathcal{C}(V_q, \mathbb{R}_{>0})$.

For $h \in \mathcal{C}_!(\phi(V_q), \mathbb{R})$ we have by left-invariance of μ

$$\mu(h) = \mu(h \circ L_g) = \int_{V_p} q_\mu(h \circ L_g \circ \phi) = \int_{V_p} q_\nu r(h \circ L_g \circ \phi)$$

and by left-invariance of ν we have

$$\begin{aligned}\mu(h) = \int_{V_q} q_\mu(h \circ \phi) = \int_{V_q} rq_\nu(h \circ \phi) &= \int_{V_q} q_\mu(r \circ \phi^{-1} \circ \phi)(h \circ \phi) \\ &= \nu((r \circ \phi^{-1})h) = \nu((r \circ \phi^{-1} \circ L_g)(h \circ L_g)) \\ &= \int_{V_p} q_\nu r_g(h \circ L_g \circ \phi)\end{aligned}$$

If $r(q) > r_g(q) = r(p)$ we can find a neighborhood V_q where $r(q) > r_g(q) + \varepsilon$ for some $\varepsilon > 0$ and $h \in \mathcal{C}_!(\phi(V_q), \mathbb{R}_{\geq 0}$ with $\nu(h) > 0$. We get a contradiction since

$$0 = \int_{V_p} q_\nu(r - r_g)(h \circ L_g \circ \phi) > \int_{V_p} \varepsilon q_\nu(h \circ L_g \circ \phi) = \varepsilon\nu(h \circ L_g) = \varepsilon\nu(h)$$

We prove now the existence of the Haar measure. The cotangent space $(T_e^*G) := (T_eG)$ is a vector space of dimension $n = \dim G$. There exists a non-zero element $\omega_e \in \Lambda^n T_e^* G$. We can make ω an invariant G-form by setting $\omega_g = L_{g^{-1}}^* \cdot \omega_e$. Here by $L_{g^{-1}} : G \to G$ induces $(L_{g^{-1}})_* : T_gG \xrightarrow{\sim} T_eG$ and on dual spaces it induces $L_{g^{-1}}^* : T_e^*G \xrightarrow{\sim} T_gG$ and also $L_{g^{-1}}^* : \Lambda^n T_e^* G \xrightarrow{\sim} \Lambda^n T_g G$.

We obtain a $\mathcal{C}^\infty$ k-form ω, i.e. a section of the line bundle $\Lambda^n T^* G$. (This means that locally on any chart ϕ the function $u \mapsto \widetilde{\omega}_{u)}) : U \to \Lambda^n \mathbb{R}^n \cong \mathbb{R}$ is $\mathcal{C}^\infty$, where $\widetilde{\omega}_{\phi(u)}$ is the image of ω_u under the isomorphism $\Lambda^n \mathbb{R}^n \cong \Lambda^n T_u^* U \cong \Lambda^n T_{\phi(u)}^* G$.

The form ω is called a volume form. The form ω is left invariant, in fact we have

$$((L_g)^*\omega)_h = L_g^* \omega_{gh} = L_g^* L_{h^{-1}g^{-1}} \omega_e = L_{h^{-1}}^* \omega_e = \omega_h.$$

Then we can use ω to obtain a Haar measure. From a volume form one gets a positive continuous density: on a chart $\phi : U \to G$ this has the density function $q_U(u) = |\phi^* \omega_{\phi(u)}(e_1, \dots e_n)|$, where $U \subset \mathbb{R}^n$ and $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. We see that since ω does not vanish, the function q_U is positive. To check that this defines a positive continuous density we check that if x is in the intersection of two charts, i.e. $x \in \phi(U) \cap \psi(V)$ we have

$$|\phi^* \omega_x(e_1, \dots e_n)| = |(\psi^{-1} \circ \phi)^* \psi^* \omega_{\psi(u)}(e_1, \dots e_n)| = |\det(d_{\phi^{-1}(x)}(\psi^{-1} \circ \phi))||\psi^* \omega_x(e_1, \dots e_n)|$$

so they glue together to a positive continuous density as discussed in (2). $\square$

We conclude this section by describing the Haar measure on $GL_n(\mathbb{R})$.

Proposition 1.9. *The Haar measure on $GL_n(\mathbb{R})$ is*

$$f \mapsto \int \frac{f(A)}{|\det(A)|^n} dA,$$

where we identify $GL_n(\mathbb{R})$ with an open subset of $\mathbb{R}^{n^2}$.

Proof. Let $\{e_{i,j}\}_{i,j\leq n}$ be the standard basis of $M_n(\mathbb{R})$. We first show

$$\omega_A = \frac{de_{1,1} \wedge de_{1,2} \wedge \ldots \wedge de_{n,n}}{\det(A)^n}$$

is a left invariant volume form on $GL_n(\mathbb{R})$.

Let $B \in GL_n(\mathbb{R})$. Recall that the linear map $L_B : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ has determinant $\det(L_B) = \det(B)^n$ (in fact, L_B can be seen as the map $(B, B, \ldots, B)$ on the columns of $M_n(\mathbb{R})$).

For $A, B \in GL_n(\mathbb{R})$ we have

$$\begin{aligned}
(L_B^*\omega)_A(e_{1,1}, \ldots, e_{n,n}) = \omega_{BA}(Be_{1,1}, \ldots, Be_{n,n}) &= \frac{Be_{1,1} \wedge Be_{1,2} \wedge \ldots \wedge Be_{n,n}}{\det(BA)^n} \\
&= \frac{\det(B)^n e_{1,1} \wedge e_{1,2} \wedge \ldots \wedge e_{n,n}}{\det(B)^n \det(A^n)} \\
&= \omega_A(e_{1,1}, \ldots, e_{n,n})
\end{aligned}$$

The formula for the Haar measure immediately follows. □

2 Maschke's theorem for compact groups

On compact groups we can choose the Haar measure so that the integral of the constant function 1 is 1.

Corollary 2.1. *Let G be a compact Lie group. Then there exists a unique Haar measure $\int_G$ on G such that $\int_G 1 = 1$, where 1_G is the constant function 1 on G.*

Proof. This is clear, because $1_G \in \mathcal{C}_!(G, \mathbb{R}) = \mathcal{C}(G, \mathbb{R})$ if G is compact. □

For $g \in G$ let $R_g : G \to G$ be the right multiplication by g.

Corollary 2.2. *Let G be a compact Lie group. Then the Haar measure is also right invariant.*

Proof. Let $\int_G$ be a Haar measure. Fix $h \in G$. Then $\Sigma_h : f \mapsto \int_G (f \circ R_h)$ is also a left invariant Haar measure, since for all $g \in G$ we have

$$\Sigma_h(f \circ L_g) = \int_G (f \circ L_g \circ R_h) = \int_G (f \circ R_h \circ L_g) = \Sigma_h(f).$$

Moreover, we have $\Sigma_h(1) = \int_G 1 \circ R_g = \int_G 1 = 1$. It follows from the uniqueness in Corollary 2.1 that $\int_G = \Sigma_h$ for all $h \in G$, hence $\int_G$ is right invariant (cf. Remark 1.7) □

Exercise 2.3. Show that if G is compact we have $\int_G f(g)dg = \int_G f(g^{-1})dg$ for all $f \in \mathcal{C}(G, \mathbb{R})$.

As a first application, we show that every representation of a compact Lie group is unitary.

Proposition 2.4. *Let V be a finite dimensional real representation of a compact Lie group G. Then there exists a G-invariant scalar product b on V, i.e the image of the homomorphism $\rho : G \to GL(V)$ is included in $O(V, b) := \{A \in GL(V) \mid b(Av, Aw) = b(v, w)\}$.*

Proof. Let $(-, -)$ be any scalar product on V. We obtain a G-invariant scalar product by setting $b(v, w) = \int_G (gv, gw)$. The scalar product b is clearly bilinear. It is positive definite since $b(v, v) = \int_G (gv, gv) > 0$ because the integral of a positive function is positive. It is left invariant because $b(hv, hw) = \int_G (ghv, ghw) = \int_G (gv, gw)$ since $\int_G$ is right invariant. □

Remark 2.5. The same proof of Proposition 2.4 shows that if V is a complex representation of G, then there exists a G-invariant hermitian product h on G. This shows that the image of $\rho : G \to GL(V)$ lies in $U(V, h) := \{A \in GL(V) \mid h(Av, Aw) = h(v, w)\}$.

Recall that a representation is said completely reducible if it is isomorphic to a direct sum of irreducible representations.

Theorem 2.6 (Maschke's theorem for compact groups)**.** *Every finite dimensional representations of a compact Lie group is completely reducible.*

Proof. Let V be a representation of a compact group G. We show by induction on $\dim V$ that V is completely reducible. The claim is clear if $\dim V = 1$.

By Proposition 2.4, we find a left G-invariant scalar product b on V. If V is irreducible, the claim is clear. Otherwise, there exists a G-stable subspace $W \subset V$. Let $W^\perp := \{v \in V \mid b(v, w) = 0 \ \forall w \in W\}$.

The subspace $W^\perp$ is also G-stable. In fact, if $g \in G$ and $v \in W^\perp$, we have for all $w \in W$ that $b(gv, w) = b(v, g^{-1}w) = 0$, so also $gv \in W^\perp$.

Since b is a scalar product we have $V = W \oplus W^\perp$. This is a decomposition into smaller representations. We can now easily conclude by induction. □

An indecomposable and completely reducible representation is by definition also irreducible, so we have the following immediate consequence.

Corollary 2.7. *A representation of a compact Lie group is indecomposable if and only if it is irreducible.*

Remark 2.8. Because of Remark 2.5, both Theorem 2.6 and Corollary 2.7 hold for complex representations. One just need to replace the G-invariant scalar product with a G-invariant hermitian product in the proof.

We conclude this section by showing how one can reconstruct a completely reducible complex representation using Schur's lemma.

Lemma 2.9 (Schur's lemma)**.** *Let V and W be irreducible complex representations of a group G. Then we have*

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \ncong W \\ \mathbb{C}\,\mathrm{Id} & \text{if } V = W \end{cases}$$

Proof. Let $f : V \to W$ be a morphism. Then $\operatorname{Ker} f$ and $\operatorname{Im} f$ are both subrepresentations. Since V and W are irreducible, if $f \neq 0$ the only possibilities are $\operatorname{Ker} f = 0$ and $\operatorname{Im} f = W$, which means that f is an isomorphism.

Assume now $V = W$. Then f is an endomorphism of V, so it has an eigenvalue $\lambda \in \mathbb{C}$. Also $f - \lambda \operatorname{Id}$ is a morphism of representations, so we must have $\operatorname{Ker}(f - \lambda \operatorname{Id}) = V$, from which it follows that $f = \lambda \operatorname{Id}$. □

Corollary 2.10. *Let V be a completely reducible complex representation of a group G. Then*

$$V \cong \bigoplus_{S \text{ irreducible}} S^{\dim_{\mathbb{C}} \operatorname{Hom}_G(S,V)}.$$

Proof. Since V is completely irreducible we can write $V \cong \bigoplus_{S \text{ irreducible}} S^{d_S}$. From Schur's lemma Lemma 2.9 we see that $\operatorname{Hom}_G(S, V) \cong \operatorname{Hom}_G(S, S^{d_S}) \cong d_S\mathbb{C}$, so $d_S = \dim_{\mathbb{C}} \operatorname{Hom}_G(S, V)$. □

We conclude this section with a remark on dual representations. Recall that if V is a representation of a group G, we can construct a dual representation V^* by setting $g \cdot \lambda(v) = \lambda(g^{-1}v)$ for all $g \in G$, $v \in V$ and $\lambda \in V^*$. This is a representation since

$$(g \cdot h \cdot \lambda)(v) = h \cdot \lambda(g^{-1}v) = \lambda(h^{-1}g^{-1}v) = gh \cdot \lambda(v).$$

If V is a complex representation of G, we can also construct the complex conjugate representation $\overline{V}$. As a vector space, $\overline{V}$ is the same as V as an additive group but scalar multiplication is defined by $z \cdot v = \overline{z}v$ for all $z \in \mathbb{C}$ and $v \in V$. The action of g on $\overline{V}$ remains the same. This defines a linear map. In fact, we have

$$g \cdot (z \cdot v) = g \cdot (\overline{z}v) = \overline{z}(g \cdot v)$$

Remark 2.11. There is a canonical isomorphism of vector space (and of representations) between $\overline{V}^* = \operatorname{Hom}_{\mathbb{C}}(\overline{V}, \mathbb{C})$ and $\overline{V^*} = \overline{\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})}$ which sends f to $\overline{f}$.

Proposition 2.12. *Let G be a compact group and let V be a finite dimensional representation of G. Then $V \cong \overline{V}^*$ if V is complex and $V \cong V^*$ if V is real.*

Proof. Assume V is complex. We can endow V with a G-invariant hermitian product $\langle -, - \rangle$. Then we can define an isomorphism $\Phi : V \to \overline{V}^*$ by $v \mapsto \langle v, - \rangle$. Notice that $\lambda(w) = \langle v, w \rangle \in \overline{V}^*$ since $\lambda(zw) = \overline{z}\langle v, w \rangle = z \cdot \lambda(w)$ and Φ is $\mathbb{C}$-linear because $\Phi(zv)(w) = \langle zv, w \rangle = z\langle v, w \rangle = z\Phi(v)(w)$. Moreover, it is a morphism of representations because

$$\Phi(gv)(w) = \langle gv, w \rangle = \langle v, g^{-1}w \rangle = (g \cdot \Phi(v))(w)$$

for any $g \in G$ and $v, w \in V$. The same proof also works in the case of real representations. □

Because of Remark 2.11 we also have an isomorphism $V \xrightarrow{\sim} \overline{V^*}$ defined by $v \mapsto \langle -, v \rangle$.

Similarly, we have an isomorphism $V^* \xrightarrow{\sim} \overline{V}$, which sends $\lambda \in V^*$ to $\lambda^\sharp \in V$ with $\lambda = \langle -, \lambda^\sharp \rangle$. We can thus define a hermitian form on V^* by setting $\langle \lambda, \mu \rangle = \langle \mu^\sharp, \lambda^\sharp \rangle = \lambda(\mu^\sharp)$.

Assume now V is an irreducible representation of G. If Then, by Lemma 2.9 the isomorphism $V \xrightarrow{\sim} \overline{V}^*$ is unique up to a scalar, so there is a unique G-invariant non-degenerate sesquilinear[1] on V is unique up to a scalar. In particular, the G-invariant hermitian product on V is unique up to a positive scalar.

[1] Sesquilinear means linear in the first component and antilinear in the second component

Let $A \in \mathrm{End}_{\mathbb{C}}(V)$. We can then consider the adjoint endomorphism $A^\dagger$ with respect to any G-invariant hermitian product, so that we have $\langle A-, -\rangle = \langle -, A^\dagger -\rangle$. Since the G-invariant hermitian product is unique up to a scalar, the adjoint map $A^\dagger$ is well defined.

Lemma 2.13. *For $A, B \in \mathrm{End}_{\mathbb{C}}(V)$, let*

$$\langle A, B\rangle_{\mathrm{End}(V)} = \mathrm{Tr}(AB^\dagger).$$

Then $\langle -, -\rangle$ is an Hermitian product on $\mathrm{End}_{\mathbb{C}}(V)$.

Proof. We need to show that $\langle A, A\rangle > 0$ if $A \neq 0$. We can choose an orthonormal basis of V. Then $A^\dagger$ is the transpose conjugate of A and we have

$$\mathrm{Tr}(AA^\dagger) = \sum_{i,j} a_{i,j} a^\dagger_{j,i} = \sum_{i,j} |a_{i,j}|^2. \qquad \square$$

3 Matrix coefficients

Definition 3.1. Let $\rho : G \to GL(V)$ be a finite dimensional representation a group G. We say that a function $c : G \to \mathbb{C}$ is a *matrix coefficient of* V if $c(g) = \lambda(\rho(g)v)$ for some $v \in V$ and $\lambda \in V^*$

Example 3.2. Let $V = \mathbb{C}^n$ with standard basis $e_1, \dots, e_n \in V$ and dual basis $e_1^*, \dots, e_n^* \in V^*$, so that $G \to GL(V) = GL_n(\mathbb{C})$. Then the function $\rho(g)_{i,j}$ which returns the (i, j)-entry of the matrix $\rho(g)$ is a matrix coefficient since $\rho(g) = e_i^*(\rho(g)e_j)$.

On the vector space of all the functions $Set(G, \mathbb{C})$ there is a left action given by $g \cdot f = f \circ (g^{-1} \cdot)$ and a right action given by $f \cdot g = f \circ (\cdot g)$. The matrix coefficients belong to $Set(G, \mathbb{C})$ and can be characterized by the following Lemma.

Lemma 3.3. *Let $c : G \to \mathbb{C}$ be a function. Then c is a matrix coefficient if and only if* $\mathrm{span}\langle c \circ (\cdot h) \mid h \in G\rangle$ *is finite dimensional.*

Proof. Assume that $c(g) = \lambda(\rho(g)v))$ is a matrix coefficient of $\rho : G \to GL(V)$. Let $v_1, \dots, v_n$ be a basis of V and let $v_1^*, \dots, v_n^*$ be the dual basis of V^*. For $h \in G$ we can write

$$\rho(h)v_i = \sum_{j=1}^{n} (\rho(h)v)_j v_j$$

for $(\rho(h)v)_j \in \mathbb{C}$. Then

$$c \circ (\cdot h)(g) = \lambda(\rho(gh)v) = \lambda(\rho(g)\rho(h)v) = \lambda(\rho(g) \sum_{j=1}^{n} (\rho(h)v)_j v_j) = \sum_{j=1}^{n} (\rho(h)v)_j \lambda(\rho(g)v_j).$$

This means that the span of all the $c \circ (\cdot g)$ is included in the vector space generated by $\lambda(\rho(g)v_j)$, for $1 \leq j \leq n$.

In the other direction, assume that $V := \mathrm{span}\langle c \circ (\cdot h) \mid h \in G\rangle$ is finite dimensional. Then V is a representation of G where the action is defined by $\rho(g)(c \circ (\cdot h) = c \circ (\cdot h) \circ (\cdot g) = c \circ (\cdot gh)$. Let $\lambda \in V^*$ defined by $\lambda(f) = f(1)$. Then we have

$$\lambda(\rho(g)c) = \lambda(c \circ (\cdot g)) = c \circ (\cdot g)(1) = c(g),$$

and so c is a matrix coefficient for V. $\square$

Remark 3.4. A similar proof shows that c is a matrix coefficient if and only if $\mathrm{span}\langle c\circ(g\cdot) \mid g \in G\rangle$ is finite dimensional.

If the group G is topological, then every matrix coefficient of a continuous representation is also continuous. So matrix coefficients of (continuous) representations are a subset of $\mathcal{C}(G,\mathbb{C}) \subset Set(G,\mathbb{C})$. We denote this set by $\mathcal{M}(G)$ The same argument as before show that a continuous function $c : G \to \mathbb{C}$ is a matrix coefficient if and only if it spans with its right (or left) translations a finite dimensional vector space.

Lemma 3.5. *The matrix coefficients $\mathcal{M}(G)$ of G form a vector space.*

Proof. It is clear that if c is a matrix coefficient and $z \in \mathbb{C}$, then zc is also a matrix coefficient. Assume that $c(g) = \lambda(g \cdot v)$ and $c'(g) = \mu(g \cdot w)$ with $v \in V$, $w \in W$, $\lambda \in V^*$ and $\mu \in W^*$.

Consider the representation $V \oplus W$ of G. Let $(\lambda, \mu) \in (V \oplus W)^*$ defined by $(\lambda, \mu)(x, y) = \lambda(x) + \mu(y)$ for all $x \in V$ and $y \in W$. Then $(c + c')(g) = (\lambda, \mu)(g \cdot (v, w))$ is also a matrix coefficient. $\square$

Proposition 3.6. *Let G be a compact Lie group. Let $c_V(g) = \lambda(g \cdot v)$ and $c_W(g) = \mu(g \cdot w)$ be two matrix coefficients, with $v \in V$, $w \in W$, $\lambda \in V^*$ and $\mu \in W^*$ where V and W are two irreducible complex representations of G.*

1. *If $V \not\cong W$, then c_V and c_W are orthogonal in $L^2(G)$, i.e. we have $\int_G c_V \overline{c_W} = 0$.*
2. *If $V = W$ then*
$$\langle c_V, c_W\rangle = \frac{\langle v, w\rangle\langle\lambda, \mu\rangle}{\dim V}.$$

Remark 3.7. Before we start the proof, we recall how to compute the trace of a rank 1 map. Let $A : V \to V$ be a linear endomorphism of rank 1 with its image is spanned by v. We can choose as basis of V the set $\{v, w_1, \dots, w_n\}$ with $\{w_i\}$ a basis of $v^\perp$ and the coefficient of v in Av is $\frac{\langle Av,v\rangle}{\langle v,v\rangle}$, so we also get $\mathrm{Tr}(A) = \frac{\langle Av,v\rangle}{\langle v,v\rangle}$.

Proof. Recall from Proposition 2.12 that $W \cong \overline{W}^* \cong \overline{(W^*)}$. Let v, w be as above and let $P : \overline{W^*} \to V$ defined by
$$P(\phi) = \int_G \overline{\phi(gw)}(gv)dg \tag{3}$$
for any $\phi \in \overline{W^*}$. (Here to compute the integral on V we choose a basis a V and simply compute the integral of the coefficients.) The map P is $\mathbb{C}$-linear: for $z \in \mathbb{C}$ we have
$$P(z \cdot \phi) = \int_G \overline{\overline{z}\phi(gw)}(gv)dg = z\int_G \overline{\phi(gw)}(gv)dg = zP(\phi).$$
Moreover, P is a morphism of representations: for any $h \in H$ we have
$$P(h \cdot \phi) = \int_G \overline{\phi(h^{-1}gw)}(gv)dg = \int_G \overline{\phi(gw)}(hgv)dg = h\left(\int_G \overline{\phi(gw)}(gv)dg\right) = h \cdot P(\phi).$$

If $V \not\cong W \cong \overline{W^*}$, then $P = 0$ by Lemma 2.9. Let $c_V(g) = \lambda(gv)$ and $c_W(v) = \mu(gw)$ be matrix coefficients with $v \in V$, $\lambda \in V^*$, $w \in W$ and $\mu \in W^*$. Then we have
$$\langle c_V, c_W\rangle = \int_G c_V(g)\overline{c_W(g)}dg = \lambda\left(\int_G \overline{\mu(gw)}(gv)dg\right) = \lambda(P(\mu)) = 0.$$

This shows the first part.

Assume now $V = W$. Consider the isomorphism $Q : V \xrightarrow{\sim} \overline{V^*}$ defined by $v \mapsto \langle v, - \rangle \mapsto \langle -, v \rangle$. The composition $P \circ Q$ is endomorphism of V, and thus $P \circ Q = t\,\mathrm{Id}$ for some $t \in \mathbb{C}$ by Lemma 2.9. To find t we compute the trace of $P \circ Q$. We have

$$P \circ Q(u) = \int_G \overline{\langle gw, u \rangle}(gv)dg = \int_G \langle u, gw \rangle (gv) dg = \int_G M_g(u) dg$$

where $M_g(u) := \langle u, gw \rangle (gv)$. Then M_g is a $\mathbb{C}$-linear endomorphism of V of rank 1 and we have $\mathrm{Tr}(M_g) = \langle gv, gw \rangle = \langle v, w \rangle$. We get

$$\mathrm{Tr}(P \circ Q) = \int_G \mathrm{Tr}(M_g) dg = \int_G \langle v, w \rangle dg = \langle v, w \rangle.$$

It follows that $P \circ Q = \frac{\langle v,w \rangle}{\dim V}$ and

$$\langle c_V, c_W \rangle = \int_G c_V(g)\overline{c_W(g)}dg = \lambda(P(\mu)) = \lambda_1(P \circ Q)(Q^{-1}\mu) = \frac{\langle v, w \rangle \lambda(\mu^\sharp)}{\dim V} = \frac{\langle v, w \rangle \langle \lambda, \mu \rangle}{\dim V}.$$

□

Corollary 3.8. *Let $\mathcal{M}(\rho) \subset \mathcal{M}(G)$ be the vector space generated by all the matrix coefficients of a representation $\rho : G \to GL(V)$. We have*

$$\bigoplus_{\rho_L \text{ irred.}} \mathcal{M}(\rho_L) \xrightarrow{\sim} \mathcal{M}(G).$$

Proof. The inclusions $\mathcal{M}(\rho_L) \subset \mathcal{M}(G)$ induce a map $\iota : \bigoplus \mathcal{M}(\rho_L) \xrightarrow{\sim} \mathcal{M}(G)$. This is surjective since every matrix coefficient is a sum of matrix coefficients for irreducible representation. Moreover, ι is injective. In fact, if $\sum_L c_L = 0$, with $c_L \in \mathcal{M}(\rho_L)$, then

$$\left\langle \sum_L c_L, \sum_L c_L \right\rangle = \sum_L \langle c_L, c_L \rangle = 0.$$

Therefore $\langle c_L, c_L \rangle = 0$ for all L, meaning that all the matrix coefficients c_L are trivial. □

There is another natural way to describe the vector space $\mathcal{M}(\rho)$ and their hermitian form.

Proposition 3.9. *Let $\mathcal{M}(\rho) \subset \mathcal{M}(G)$ be the vector space generated by all the matrix coefficients of a representation $\rho : G \to GL(V)$. If ρ is irreducible, we have an isomorphism*

$$\begin{aligned} M : \mathrm{End}(V) &\xrightarrow{\sim} \mathcal{M}(\rho) \\ A &\mapsto (c_A(g) = \mathrm{Tr}(\rho(g)A)) \end{aligned}$$

and $\sqrt{\dim V}\, M$ is an isometry.

Proof. Recall that $\mathrm{End}(V) \cong V^* \otimes V$, where $\lambda \otimes v$ correspond to the rank 1 linear map $w \mapsto \lambda(w)v$. The map $(\lambda, v) \mapsto (g \mapsto \lambda(\rho(g)v)$ is bilinear, so it extends to a surjective linear map $\mathrm{End}(V) \to \mathcal{M}(\rho)$.

By linearity, it is enough to check that $M(A)(g) = \dim(V)\,\mathrm{Tr}(\rho(g)A)$ if A is of rank 1. So we assume $A(w) = \lambda(w)v$ for some $\lambda \in V^*$ and $v \in V \setminus \{0\}$. Since the image of $\rho(g)A$ is spanned by gv, we have

$$\mathrm{Tr}(\rho(g)A) = \frac{\langle g(A(gv)), gv \rangle}{\langle gv, gv \rangle} = \frac{\langle A(gv), v \rangle}{\langle v, v \rangle} = \lambda(gv).$$

Let now B a rank 1 endomorphism of the form $B = \mu(-)w$, with $w \in V$ and $\mu \in V^*$. Then its adjoint is $B^\dagger = \langle -, w\rangle \mu^\sharp$. In fact, we have

$$\langle Bv_1, v_2\rangle = \mu(v_1)\langle w, v_2\rangle = \langle w, v_2\rangle\langle v_1, \mu^\sharp\rangle = \langle v_1, \overline{\langle w, v_2\rangle}\mu^\sharp\rangle = \langle v_1, B^\dagger v_2\rangle.$$

So, if A and B are of rank 1 as above, we have

$$\langle A, B\rangle_{\mathrm{End}(V)} = \mathrm{Tr}(AB^\dagger) = \frac{\langle AB^\dagger v, v\rangle}{\langle v, v\rangle} = \frac{\langle\langle v, w\rangle A\mu^\sharp, v\rangle}{\langle v, v\rangle} = \langle v, w\rangle\langle\lambda, \mu\rangle$$

It follows that $\sqrt{\dim V}M$ is an isometry and in particular an isomorphism. □

Remark 3.10. The isomorphism M is also an isomorphism of $G \times G^{op}$ representations. In fact, both on $\mathrm{End}(V)$ and on $\mathcal{M}(\rho)$ we have a left and a right G-action and for any $g, h \in G$ we have

$$M(\rho(h_1)A\rho(h_2))(g) = \mathrm{Tr}(\rho(g)\rho(h_1)A\rho(h_2)) = \mathrm{Tr}(\rho(h_2gh_1)A) = M(A)(h_2gh_1).$$

4 Compact operators and the spectral theorem*

In this section we discuss the spectral theorem for compact operators. This theorem will be applied to prove Peter–Weyl theorem for compact (Lie) groups. However, the arguments are not geometric nor group theoretic but rather belong to the theory of Hilbert spaces in functional analysis.

Let $\mathcal{H}$ be a Hilbert space, that is a real (or complex) vector space with a scalar (or hermitian) product $\langle -, -\rangle$ which makes it a complete topological space.

Example 4.1. If G is a compact Lie group, we consider the space of square integrable functions $L^2(G) := \{f : G \to \mathbb{C} \mid \int_G |f(g)|^2 < \infty\}$, where $\int_G$ is the Haar measure. This is a Hilbert space with respect to the hermitian product is $\langle f_1, f_2\rangle = \int_G f_1(g)\overline{f_2(g)}dg$. On an open set $U \subset \mathbb{R}^n$ every function $f \in L^2(G)$ can be approximated with $\mathcal{C}^\infty$-functions with compact support [4, Satz 2.6.1]. In particular, the space $\mathcal{C}^\infty(G) \subset L^2(G)$ is dense.

However, for our purposes, is enough to simply define $L^2(G)$ as the completion of the space $\mathcal{C}(G)$ with respect to the metric $\langle -, -\rangle$ as above.

Definition 4.2. We say that a linear map $T : \mathcal{H} \to \mathcal{H}$ is *self-adjoint* if $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in \mathcal{H}$.

The norm of a linear map T is defined as $\|T\| := \sup\{\|T(v)\| \mid v \in \mathcal{H} \text{ and } \|v\| = 1\}$. A linear map is continuous if and only if it has finite norm.

Definition 4.3. We say that T is *compact* if for any bounded sequence (v_i) in $\mathcal{H}$, the sequence (Tv_i) has a converging subsequence. Notice that a compact operator has finite norm.

Lemma 4.4. *Let $\mathcal{H} \neq 0$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a compact and self-adjoint operator. Then one between $\|T\|$ and $-\|T\|$ is an eigenvalue of T.*

Proof. We have $\|T^2\| \leq \|T\|^2$ since for any linear map T since $\|T^2(v)\| \leq \|T\|\|T(v)\| \leq \|T\|^2\|v\|$. Since T is self-adjoint, we also have $\|T^2\| \geq \|T\|^2$. In fact, for any $v \in \mathcal{H}$ of norm 1. We have $\|T(v)\|^2 = \langle Tv, Tv\rangle = \langle v, T^2(v)\rangle \leq \|v\|\|T^2(v)\|$. It follows that $\|T^2\| = \|T\|^2$.

Let now v_n be a sequence of vectors of norm 1 such that $\lim \|T^2 v_n\| = \|T^2\|$. Since T is compact, up to replacing with a subsequence, we can assume that the sequences Tv_n and $T^2 v_n$ converge.

Since $\|T^2 v_n\| \leq \|T\|\|Tv_n\|$, we have $\|T\| = 0$, in which case $T = 0$ and the statement is clear, or $\|T\| \geq \lim \|Tv_n\| \geq \frac{\|T^2\|}{\|T\|} = \|T\|$, and so $\lim \|Tv_n\| = \|T\|$.

Consider the sequence $a_n := \|(T^2 - \|T\|)v_n\|$. Since T is self-adjoint, we have

$$a_n = \langle v_n, (T^4 - 2\|T\|^2 T^2 + \|T^4\|)v_n\rangle = \|T^2 v_n\|^2 - 2\|T\|^2\|Tv_n\|^2 + \|T\|^4$$

and $\lim_{n\to\infty} a_n = 0$.

Let $w = \lim T^2 v_n$. Then we obtain also $\lim \|T\|^2 v_n = w$, and since we can assume $\|T\| \neq 0$, also v_n converges to $v := \frac{w}{\|T\|}$. Then we have $T^2 v = \lim T^2 v_n = \lim \|T^2\| v_n = \|T\|^2 v$. It follows that

$$(T - \|T\|)(T + \|T\|)(v) = 0,$$

so either $Tv = -\|T\|v$ or $Tv' = \|T\|v'$ for $v' = (T + \|T\|)(v)$. □

For a linear map $T : \mathcal{H} \to \mathcal{H}$, we denote by $\mathcal{H}_\lambda$ the eigenspace associated to the eigenvalue $\lambda \in \mathbb{C}$. The eigenspaces are in direct sum, but in general we only have an inclusion $\bigoplus \mathcal{H}_\lambda \subset \mathcal{H}$.

Lemma 4.5. *If T is self-adjoint, then the eigenspaces of T are orthogonal to each other.*

Proof. Let $v \in \mathcal{H}_\lambda$ and $w \in \mathcal{H}_\mu$ with $\mu \neq \lambda$. Then $\lambda\langle v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \mu\langle v, w\rangle$, from which it follows $\langle v, w\rangle = 0$. □

Let $\mathcal{H}$ be a Hilbert space, and let $V_i \subset \mathcal{H}$, for $i \in I$, be orthogonal subspaces. We write $\widehat{\bigoplus}_{i\in I} V_i$ for the closure of the vector space $\bigoplus_{i\in I} V_i$. Recall that a subspace is dense if and only if its orthogonal is trivial.

Theorem 4.6 (Spectral theorem for compact operators)**.** *Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a compact and self-adjoint operator. Then $\mathcal{H} = \widehat{\bigoplus}_{\lambda\in\mathbb{C}} \mathcal{H}_\lambda$ and all the eigenspaces $\mathcal{H}_\lambda$ for $\lambda \neq 0$ are finite dimensional.*

Proof. Assume that the direct sum is not dense, so it has a non-trivial orthogonal W. Then T restricts to a compact operator on W. In fact, if $w \in W$, we have $\langle Tw, v\rangle = \langle w, Tv\rangle = \lambda\langle w, v\rangle = 0$ for all $v \in \mathcal{H}_\lambda$. But Lemma 4.4, shows that T has an eigenvector in W, which gives a contradiction since $\langle v, v\rangle > 0$.

Assume that $\mathcal{H}_\lambda$ is infinite dimensional for $\lambda \neq 0$. Then we can find a sequence of vectors v_n of norm 1, for example an orthonormal basis, which does not converge, such that also $Tv_n = \lambda v_n$ does not converge. □

5 Convolutions on compact Lie groups

Let G be a compact Lie group with Haar measure $\int_G$. On $\mathcal{C}(G)$ we can consider several different norms.

$$\|f\|_1 := \int_G |f(g)|, \quad \|f\|_2 = \sqrt{\int_G |f(g)|^2}, \quad \|f\|_\infty = \sup_{g\in G} f(g).$$

For any $f \in \mathcal{C}(G)$ we have $\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$. In fact, $\|f\|_1 = \int_G |f| = \langle |f|, 1\rangle \leq \|f\|_2 \cdot \|1\| = \|f\|_2$, while the second inequality is trivial. In particular, we have $\mathcal{C}(G) \subset L^\infty(G) \subset L^2(G) \subset L^1(G)$.

Definition 5.1. If $\phi \in \mathcal{C}(G)$ and $f \in L^2(G)$ we define the convolution

$$T_\phi(f)(g) := (\phi * f)(g) := \int_G \phi(gh^{-1})f(h)dh.$$

Remark 5.2. Using the change of variable $h \mapsto h^{-1}g$ (which we can because the Haar measure is right invariant and inv-invariant), we also get

$$(\phi * f)(g) = \int_G \phi(h)f(h^{-1}g)dh.$$

The goal of this section is to show that convolution with a continuous function gives a compact operator on $L^2(G)$. We start by proving a version of the Heine–Cantor theorem for compact Lie groups.

Proposition 5.3. *Let G be a compact Lie group and let $f \in \mathcal{C}(G)$. Then for any ε there exists a neighborhood U of the unity $e \in G$ such that for any $g \in G$ and $u \in U$ we have $|f(g) - f(ug)| < \varepsilon$.*

Proof. Let $\psi : U \to G$ be a local chart around $e \in G$ where U is an open set in $\mathbb{R}^{\dim G}$ containing 0 and let $B(r)$ be the image under ψ of the open ball around $0 \in U$ of radius r.

Fix $\varepsilon > 0$. For any $g \in G$ by continuity there is $\delta_g > 0$ such that for any $u \in B_{\delta_g}$ we have $|f(g) - f(ug)| < \frac{\varepsilon}{2}$. Then $\bigcup_{g \in G} B(\frac{\delta_g}{2})g$ is a open cover of G and by compactness it admits a finite subcover $\bigcup_{i=1}^k B(\frac{\delta_{g_i}}{2})g_i$. Let δ' be the minimum among the δ_{g_i}. Since the multiplication is continuous, we can find $\delta > 0$ such that $B(\delta) \cdot B(\delta) \subset B_{\delta'} \subset B(\delta_{g_i})$ for any i.

Let now $g \in G$ and $u \in B_\delta$. We have $g \in B(\frac{\delta_{g_i}}{2})g_i$ for some i. Then also $ug \in B(\delta_{g_i})g_i$ because $ugg_i^{-1} \in B(\delta)B(\delta) \subset B(\delta_{g_i})$. We have

$$|\phi(ug) - \phi(g)| \leq |\phi(ug) - \phi(g')| + |\phi(g') - \phi(g)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

We are also going to need the Arzelà–Ascoli theorem, which we recall.

Theorem 5.4 (Arzelà–Ascoli)**.** *Let X be a compact topological space and let $B \subset \mathcal{C}(X)$ a bounded and equicontinuous set, i.e. for any $x \in X$ and $\varepsilon > 0$ there exists a neighborhood U of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U$ and $f \in B$. Then every sequence in B has a uniformly convergent subsequence.*

Proof. Fix $\varepsilon > 0$. For any x there is a neighborhood U_x of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U_x$ and $f \in B$. There exists finitely many $x_1, \dots, x_r$ such that $U_{x_1}, \dots, U_{x_r}$ cover X.

Since B is bounded, there exists $M \in \mathbb{Z}_{>0}$ such that $-M\varepsilon < f(x) < M\varepsilon$ for all $x \in X$ and $f \in B$.

Let $\sigma : \{1, 2, \dots, r\} \to \{-M, -M+1, \dots, M-1, M\}$ be an arbitrary function. It there exists such a function, let $f_\sigma \in B$ such that $\sigma(i)\varepsilon \leq f_\sigma(x_i) < (\sigma(i) + 1)\varepsilon$ for all i.

We claim that for any $f \in B$ there exists σ such that $\|f - f_\sigma\|_\infty < 3\varepsilon$. In fact, we can choose σ such that $\sigma(i)\varepsilon \leq f(x_i) < (\sigma(i) + 1)\varepsilon$ (so f_σ exists!). For any $x \in X$ we have $x \in U_{x_i}$ for some i and

$$|f(x) - f_\sigma(x)| \leq |f(x) - f(x_i)| + |f(x_i) - f_\sigma(x_i)| + |f_\sigma(x_i) - f_\sigma(x)| < 3\varepsilon.$$

Let now f_n be a sequence in B. For any $m > 0$ we can find a function σ_m as before such that there are infinitely many functions f_n in the ball $B(f_{\sigma_m}, 3^{-m})$. So we can find subsequences $(f_n^{(1)}) \subset (f_n)$ in $B(f_{\sigma_1}, 3^{-1})$, $(f_n^{(2)}) \subset (f_n^{(1)})$ in $B(f_{\sigma_2}, 3^{-2})$ and so on. Finally choose $f_{n_1} \in (f_n^{(1)})$, $f_{n_2} \in (f_n^{(2)})$ and so on.

The sequence f_{n_k} is a Cauchy sequence and it is therefore uniformly convergent. $\square$

Proposition 5.5. *Let $\phi \in \mathcal{C}(G)$ and consider the linear operator $T_\phi : L^2(G) \to L^2(G)$ defined by $T_\phi(f) = \phi * f$. Then T_ϕ is a compact operator and the image of T_ϕ is contained in $\mathcal{C}(G)$.*

Proof. The function ϕ is uniformly continuous and by Proposition 5.3, for any $\varepsilon > 0$ we can find a neighborhood U of the unity $e \in G$ such that $|\phi(g) - \phi(ug)| < \varepsilon$ for any $g \in G$ and $u \in U$. For any $f \in L^2(G)$ and $g \in G$ and $u \in U$ we have

$$\begin{aligned} |T_\phi f(g) - T_\phi f(ug)| &= \left| \int_G (\phi(gh^{-1}) - \phi(ugh^{-1})) f(h) dh \right| \\ &\leq \int_G \left| \phi(gh^{-1}) - \phi(ugh^{-1}) \right| \cdot |f(h)| dh \leq \varepsilon \|f\|_1 \leq \varepsilon \|f\|_2 \end{aligned} \tag{4}$$

Notice that this implies that $T_\phi f$ is continuous, and so it is in $L^2(G)$.

To show that T_ϕ is compact, we need to show that every succession in $B := \{T_\phi(f) \mid f \in L^2(G), \|f\|_2 \leq 1\}$ has a converging subsequence. This follows from the Arzelà–Ascoli theorem after we show that B is bounded and equicontinuous. It is bounded because

$$\begin{aligned} \|T_\phi(f)\|_2 \leq \|T_\phi(f)\|_\infty &= \sup_{g \in G} \left| \int_G \phi(gh^{-1}) f(h) dh \right| \\ &\leq \sup_{g \in G} \|\phi\|_\infty \int_G |f(h)| dh = \|\phi\|_\infty \|f\|_1 \leq \|\phi\|_\infty \|f\|_2. \end{aligned} \tag{5}$$

It is equicontinuous because by (4) we have $|T_\phi f(g) - T_\phi f(ug)| \leq \varepsilon \|f\|_2 \leq \varepsilon$ for any $f \in B$. We have then equicontinuity by taking the neighborhood Ug of $g \in G$. $\square$

We need a final ingredient to apply the spectral theorem: self-adjointness.

Proposition 5.6. *Assume that $\phi(g^{-1}) = \overline{\phi(g)}$ for all $g \in G$. Then $T_\phi : L^2(G) \to L^2(G)$ is self-adjoint.*

Proof. Let $f_1, f_2 \in L^2(G)$. Then, by Fubini's theorem [4, Satz 1.7.16], we have

$$\langle T_\phi f_1, f_2 \rangle = \int_G \left(\int_G \phi(gh^{-1}) f_1(h) dh \right) \overline{f_2(g)} dg = \int_G \int_G \phi(gh^{-1}) f_1(h) \overline{f_2(g)} dg dh$$

$$\langle f_1, T_\phi f_2 \rangle = \int_G f_1(g) \overline{\left(\int_G \phi(gh^{-1}) f_2(h) dh \right)} dg = \int_G \int_G f_1(g) \phi(hg^{-1}) \overline{f_2(h)} dg dh$$

and the equality $\langle T_\phi f_1, f_2 \rangle = \langle f_1, T_\phi f_2 \rangle$ follows by swapping the role of g and h. $\square$

6 Peter–Weyl theorem

We can now prove the first part of Peter–Weyl theorem

Theorem 6.1 (Peter–Weyl theorem - First version)**.** *Let G be a compact Lie group. Then the matrix coefficients are dense in $\mathcal{C}(G)$ with respect to the norm $\| - \|_\infty$.*

Proof. Let $f \in \mathcal{C}(G)$ be a continuous function on G. Since G is compact, by Proposition 5.3 there exists a neighborhood U of $e \in G$ such that for any $g \in G$ and $u \in U$ we have $|f(u^{-1}g) - f(g)| < \varepsilon$.

Let now $0 \neq \psi \in \mathcal{C}_!(U \cap U^{-1}, \mathbb{R}_{\geq 0})$. Let $\phi(g) := \psi(g) + \psi(g^{-1})$. Then also $\phi \in \mathcal{C}_!(U \cap U^{-1}, \mathbb{R}_{\geq 0})$. Up to rescaling we can assume $\int_G \phi = 1$. Notice that we have $\phi(g) = \phi(g^{-1})$ for all $g \in G$. Therefore, by Proposition 5.6, the operator T_ϕ is self-adjoint on $L^2(G)$, and we have

$$|T_\phi f(g) - f(g)| = \left| \int_G \phi(h) f(h^{-1}g) - \phi(h) f(g) dh \right| \leq \int_G \phi(h) |f(h^{-1}g) - f(g)| dh \leq \varepsilon$$

for all $g \in G$. Let $L^2(G)_\lambda$ be the eigenspaces of T_ϕ. Since T_ϕ is compact and self-adjoint, we have

$$L^2(G) = \widehat{\bigoplus}_\lambda L^2(G)_\lambda,$$

so for any ε' there exists $p \in \bigoplus L^2(G)_\lambda$ approximating f with $\|f - p\|_2 < \varepsilon'$. We choose $\varepsilon' = \frac{\varepsilon}{\|\phi\|_\infty}$. We have by (5) that

$$\|T_\phi f - T_\phi p\|_\infty = \|T_\phi(f - p)\|_\infty \leq \|\phi\|_\infty \|f - p\|_2 \leq \varepsilon.$$

So it follows that $\|f - T_\phi p\|_\infty < 2\varepsilon$.

Moreover, we have $T_\phi p \in \bigoplus_{\lambda \neq 0} L^2(G)_\lambda$. In particular, it is included in a finite direct sum of finite dimensional vector spaces.

Claim 6.2. If $\lambda \neq 0$, every element of $L^2(G)_\lambda$, for $\lambda \neq 0$, consists of a matrix coefficients.

Proof of the claim. Let $f \in L^2(G)_\lambda$. For $g, x \in G$ we have

$$T_\phi(f \circ (\cdot g)(x)) = \int_G \phi(xh^{-1}) f(hg) dh = \int_G \phi(xgh^{-1}) f(h) dh = (T_\phi f)(gx) = \lambda (f \circ (\cdot g))(x).$$

since $\int_G$ is right invariant. Since $L^2(G)_\lambda$ is finite dimensional, the statement follows from Lemma 3.3. □

It follows that $T_\phi p$ is a is a matrix coefficient. We have thus successfully approximated any function f in $\mathcal{C}(G)$ with a a matrix coefficient. □

Since the continuous function are dense in L^2 (see for example [4, Satz 2.6.1]) we have the following consequence.

Corollary 6.3. *Let G be a compact Lie group. Then the matrix coefficients are dense in $L^2(G)$ with respect to the norm $\| - \|_2$.*

7 Fourier theory of compact group

We prove now a second version of the Peter–Weyl theorem, which usually also referred to as non-abelian Fourier analysis, since it is a natural generalization of the Fourier series (see Example 8.4).

Recall that if $\rho : G \to GL(V)$ is a representation of G we denote by $\mathcal{M}(\rho)$ the vector space of matrix coefficients of ρ and and by $\mathcal{M}(G)$ the vector space of all matrix coefficients.

Then we have a surjective morphism

$$\mathcal{G} : \bigoplus_{L \text{ irred.}} \operatorname{End}_{\mathbb{C}}(L) \to \mathcal{M}(G)$$
$$A \in \operatorname{End}_{\mathbb{C}}(L) \mapsto \sqrt{\dim L}\, c_A$$

We have already shown in Proposition 3.9, that $\mathcal{F}$ is injective and it is surjective since every matrix coefficient can be written as sum of matrix coefficients for irreducible representations (cf. Lemma 3.5). So $\mathcal{F}$ is an isomorphism (of $G \times G^{op}$-representations, cf. Remark 3.10) Moreover, it is an isometry by Proposition 3.6 if we endow each $\mathrm{End}_{\mathbb{C}}(L)$ with the scalar product $(A, B) = \dim(L)\,\mathrm{Tr}(AB^{\dagger})$.

Let $f \in \mathcal{C}(G)$ be a continuous function. Then on any representation V of G, f induces a linear map $\widehat{f}_V : V \to V$ defined by $\widehat{f}_V(v) = \int_G f(g^{-1})gvdg$.[2]

We can give a second version of the Peter–Weyl theorem, also known as the Fourier series expansion for compact groups.

Theorem 7.1. *The morphism $\mathcal{G}$ extends to an isomorphism of Hilbert spaces*

$$\mathcal{G} : \widehat{\bigoplus_{L\ irred.}} \mathrm{End}_{\mathbb{C}}(L) \to L^2(G)$$

and the inverse $\mathcal{F} := \mathcal{G}^{-1}$ sends $f \in L^2(G)$ to the tuple of endomorphism $\sqrt{\dim L}\ \widehat{f}_L : L \to L)$.

Proof. We have an isometry, so it is an isomorphism of pre-Hilbert spaces and it extends to an isomorphism on the completions. By Corollary 6.3 we know that the completion of $\mathcal{M}(G)$ is $L^2(G)$. It remains to compute the inverse of $\mathcal{G}$.

Since $\mathcal{M}(G)$ is dense in $\mathcal{C}(G)$, we can assume $f \in \mathcal{M}(G)$. In fact if $f_n \to f$ in L^2 then $\widehat{f_n}_V \to \widehat{f}_V$ since on compact spaces $\|f - f_n\|_1 \le \|f - f_n\|_2$. By linearity we can assume $f(g) = \phi(gv)$ for some $w \in W$ and $\phi \in W^*$, with V irreducible. We need to show that $\mathcal{F}(f)$ is the endomorphism $(\sqrt{\dim V})^{-1}\phi(-)w : W \to W$

Then, for an irreducible representation V and $v \in V$ we have

$$\widehat{f}_V v = \int_G \phi(g^{-1}w)gvdg = \int_G \langle w, g\phi^{\sharp}\rangle gvdg = \int_G \overline{\langle g\phi^{\sharp}, w\rangle} gvdg = P(\langle -, w\rangle) = P \circ Q(w)$$

with P defined similarly as in (3) and $Q(w) = \langle -, w\rangle$. So

$$\mathcal{F}(f)(v) = \sqrt{\dim V}\,\widehat{f}_V(v) = \begin{cases} 0 & \text{if } V \ncong W \\ \sqrt{\dim V}\ P \circ Q(w) = \frac{\langle v, \phi^{\sharp}\rangle}{\sqrt{\dim V}} w = \frac{\phi(v)w}{\sqrt{\dim V}} & \text{if } V = W. \end{cases} \qquad \square$$

The isomorphism $\mathcal{F}$ is also compatible with the convolution product when this is defined. However, notice that in general $L^2(G)$ is not an algebra since the convolution of two L^2-functions may be not in $L^2(G)$.

We define a new operation on $\bigoplus \mathrm{End}_L(\mathbb{C})$, by setting $A \star B = \frac{1}{\sqrt{\dim L}} A \circ B$ for any $A, B \in \mathrm{End}(L)$

Proposition 7.2. *Let $f_1, f_2 \in \mathcal{C}(G)$ we have*

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2) \star \mathcal{F}(f_1).$$

Proof. Let V be an irreducible representation and let $v \in V$. We have

$$\mathcal{F}(f_1 * f_2)(v) = \sqrt{\dim V} \int_G f_1 * f_2(g^{-1})gvdg = \sqrt{\dim V} \int_G \int_G f_1(g^{-1}h^{-1}) f_2(h) gvdhdg.$$

[2] This notation is probably non-standard. Usually one defines $\widehat{f}_V(v) = \int_G f(g)gvdg$.

On the other hand we have

$$\begin{aligned}
\mathcal{F}(f_2) \star \mathcal{F}(f_1)(v) &= \mathcal{F}(f_2)\left(\int_G f_1(g^{-1})(gv)dg\right) \\
&= \sqrt{\dim V}\int_G f_2(h^{-1})\left(\int_G f_1(g^{-1})(hgv)dg\right)dh \\
&= \sqrt{\dim V}\int_G\int_G f_1(g^{-1})f_2(h)(h^{-1}gv)dgdh \\
&= \sqrt{\dim V}\int_G\int_G f_1(g^{-1}h^{-1})f_2(h)gvdgdh.
\end{aligned}$$

where we have used that we can change h with h^{-1} in the third equality and that $\int_G$ is invariant under left multiplication with h^{-1}. $\square$

$\chi_V * \chi_W = 0$ if V and W are non-isomorphic and $\chi_V * \chi_V = \mathcal{G}(\mathcal{G}(\chi_V * \chi_W)$
We have

$$\chi_L * \chi_L(g) = \int_G \chi_L(gh^{-1})\chi_L(h)dh = \int_G \operatorname{Tr}(\rho(gh^{-1})\operatorname{Tr}(\rho(h))dh$$

$\chi_L * \chi_L = \chi_L$

Corollary 7.3. *Let G be a compact group and let $\{\rho_L : G \to U(d_L, \mathbb{C})\}_{L\in I}$ be a set of representatives of the isomorphism classes of the irreducible representations of G. Then the matrix coefficients $\{\sqrt{\dim L}(\rho_L)_{i,j}\}_{\substack{L\in I\\ i,j\leq \dim L}}$ form an Hilbert basis of $L^2(G)$.*

Proof. The morphism $\mathcal{G}$ is an isometry, and the matrix coefficients $\sqrt{\dim L}(\rho_L)_{i,j}$ are precisely the image of the elementary matrices $E_{i,j}$, which form an orthonormal basis of $\operatorname{End}(\mathbb{C}^{\dim L}) \cong M_{n\times n}(\mathbb{C}^{d_L})$. $\square$

8 Peter–Weyl for Class functions

The image of $\mathrm{Id}_L \in \operatorname{End}_{\mathbb{C}}(L)$ under this isomorphism is $\sqrt{\dim L}\chi_L(g)$ where $\chi_L(g) := \operatorname{Tr}(\rho_L(g))$. We call the trace of an irreducible representation a *character* of G.

If V and W are irreducible representations, by Proposition 7.2 we have

$$\begin{aligned}
\chi_V * \chi_W &= \mathcal{G}(\mathcal{F}(\chi_V * \chi_W)) = \mathcal{G}(\mathcal{F}(\chi_V) \star \mathcal{F}(\chi_W)) \\
&= \mathcal{G}\left(\frac{\mathrm{Id}_V \star \mathrm{Id}_W}{\sqrt{\dim V \cdot \dim W}}\right) = \begin{cases} 0 & \text{if } V \ncong W \\ \frac{\chi_V}{\dim V} & \text{if } V = W. \end{cases}
\end{aligned}$$

Let now $e_L := (\dim L)\chi_L$. Then we have $e_L * e_L = e_L$.

Definition 8.1. A *class function* is a function $f : G \to \mathbb{C}$ is a function that it is constant of the conjugacy classes of G, i.e. such that for any $g, h \in G$ we have $f(g^{-1}hg) = f(h)$.

Clearly, the character functions χ_L are class functions since the trace is invariant under conjugation.

Proposition 8.2. *Let G be a compact Lie group. Then the irreducible characters of G generate a dense subspace of the space of continuous class functions on G.*

Proof. Let ϕ be a class function. Thanks to Theorem 6.1. we can find a matrix coefficient $c \in \mathcal{M}(G)$ such that $\|\phi - c\|_\infty < \varepsilon$. Consider the function

$$\widetilde{c}(x) := \int_G c(gxg^{-1})dg.$$

This is a class function, since $\widetilde{c}(h^{-1}xh) = \int_G c(ghxh^{-1}g^{-1})dg = \int_G c(gxg^{-1})dg$ and we have

$$|\phi(x) - \widetilde{c}(x)| = \left| \int_G (\phi(gxg^{-1}) - c(gxg^{-1}))dg \right| \leq \|\phi - c\|_\infty < \varepsilon.$$

Since G is compact, we can write $c = \sum c_i$ as a finite sum of matrix coefficients for irreducible representations V_i with $c_i(g) = \lambda_i(gv_i)$, for $v \in V_i$ and $\lambda \in V_i^*$, we have

$$\widetilde{c_i}(x) = \int_G \lambda_i(gxg^{-1}v_i)dg = \lambda_i \left(\int_G gxg^{-1}v_i dg \right)$$

Consider the morphism $A_x^i : V_i \to V_i$ defined by $v \mapsto \int_G gxg^{-1}vdg$. We have

$$A_x^i(hv) = \int_G gxg^{-1}hvdg = \int_G hh^{-1}gxg^{-1}hvdg = hA_x^i(v).$$

So A is G-equivariant, and by Schur's Lemma 2.9 we get $A = r_x^i \operatorname{Id}_{V_i}$ for some $r_x^i \in \mathbb{C}$. Computing the trace we obtain

$$r_x^i \dim V_i = \operatorname{Tr}(A_x^i) = \int_G \operatorname{Tr}(\rho(gxg^{-1})dg = \int_G (\operatorname{Tr}(\rho(x)) = \chi_{V_i}(x),$$

so

$$\widetilde{c}(x) = \sum \widetilde{c_i}(x) = \sum \lambda_i(r_x^i v_i) = \sum \frac{\lambda_i(v_i)}{\dim V_i} \chi_{V_i}(x)$$

and thus it is a linear combination of characters. □

Corollary 8.3. *For any class function $f \in L^2(G)$ we have*

$$f = \sum_{L \text{ irred.}} \langle f, \chi_L \rangle \chi_L,$$

with the series converging with respect to $\| - \|_2$ and

$$\|f\|_2^2 = \sum_{L \text{ irred.}} \langle f, \chi_L \rangle^2.$$

Proof. If f is a class function, then also $\widehat{f}_L$ is invariant under conjugation. In fact for $v \in L$ we have

$$h\widehat{f}_L h^{-1}v = \int_G f(g^{-1})(hgh^{-1}v)dg = \int_G f(hg^{-1}h^{-1})gvdg = \int_G f(g^{-1})gvdg = \widehat{f}_L(v)$$

So $\widehat{f}_L(hv) = h\widehat{f}_L(v)$ and $\widehat{f_L}$ is a morphism of representations. If L is irreducible, by Schur's Lemma 2.9 we must have $\widehat{f}_L = \alpha \operatorname{Id}_L$ for some $\alpha \in \mathbb{C}$. To find α we compute as usual the trace of $\widehat{f}_L$. We have

$$\operatorname{Tr}(\widehat{f}_L) = \int_G f(g^{-1} \operatorname{Tr}(\rho_L(g))dg = \int_G f(g^{-1})\chi_L(g)dg = \int_G f(g)\chi_L(g^{-1}) = \langle f, \chi_L \rangle,$$

where we used that $\chi_L(g^{-1}) = \overline{\chi_L(g)}$ since we can always find a basis on which $\rho(g)$ is unitary. So $\widehat{f}_L = \frac{\langle f, \chi_L \rangle}{\dim L} \operatorname{Id}_L$ and we conclude that

$$f = \mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(\frac{1}{\sqrt{\dim L}} \sum_L \langle f, \chi_L \rangle \operatorname{Id}_L) = \sum_L \langle f, \chi_L \rangle \chi_L. \qquad \square$$

Example 8.4. If the group G is abelian, then every function is a class function and moreover all the irreducible representations are of dimension one. So one gets that every L^2-function can be approximated with the characters of the irreducible representations $G \to \mathbb{C}^*$.

In the case of S^1 we recover the classical results from Fourier analysis: if $f \in L^2(S^1)$, then

$$f = \sum_{n \in \mathbb{Z}} \langle f, e^{in\theta} \rangle e^{in\theta} = \frac{1}{2\pi} \left(\int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta \right) e^{in\theta}.$$

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