Lie Groups
SoSe 2023 — Ubungsblatt 1

26.04.2023

---

1 Topological groups

Aufgabe 1.1: Let G be a topological group and let H be a subgroup. Show that H is a topological group with the subspace topology.
Let N be a normal subgroup. Show that G/N is a topological group with the quotient topology.

Aufgabe 1.2: Let G be a topological group and let H be a subgroup of G.

1. Show that H is open if and only if it contains a neighborhood of the identity $e \in G$.
2. Show that if H is open, then H is also closed.
3. Show that if G is connected and H is open then $G = H$.

Bonus-Aufgabe 1.3: Let $m, n \in \mathbb{Z}$ with $m \neq 0$. We define $d_5(m,n) = \frac{1}{5^k}$ where k is the maximal integer such that 5^k divides $m - n$. If $m = n$ let $d_5(m,n) = 0$. Then d_5 defines a metric on $\mathbb{Z}$ called the 5-adic metric (one can do the same for every prime p).
Show that $(\mathbb{Z}, +)$ is a topological group with respect to the topology induced by the metric d_5.

2 Representation theory of S^1

Recall: Representations of S^1 are assumed to be continuous!

Aufgabe 1.4: For $n, m \in \mathbb{Z}$ let $\rho_{n,m} : S^1 \times S^1 \to GL_1(\mathbb{C}) \cong \mathbb{C}^\times$ be the map defined by $\rho_{m,n}(z, z') = z^m (z')^n$.

1. Show that $\rho_{m,n}$ defines a complex representation of $S^1 \times S^1$.
2. Show that $\rho_{m,n}$ is isomorphic to $\rho_{m',n'}$ (as a representation) if and only if $m = m'$ and $n = n'$.
3. Show that every representation of $S^1 \times S^1$ on $\mathbb{C}$ is isomorphic to $\rho_{n,m}$ for some $n, m \in \mathbb{Z}$.

Aufgabe 1.5: Show that every irreducible real representation of S^1 is either the trivial representation or is a 2-dimensional representation isomorphic to

$$\rho_n : S^1 \to GL_2(\mathbb{R})$$
$$e^{i\theta} \mapsto \begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix}.$$

for some $n \geq 1$.