Lie Groups
SoSe 2023 — Ubungsblatt 2

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Matrix groups and exponential

Aufgabe 2.1: Let $SO_2(\mathbb{R}) = O_2(\mathbb{R}) \cap SL_2(\mathbb{R}) = \{A \in Mat_2(\mathbb{R}) \mid AA^t = Id$ and $\det(A) = 1\}$. Show that $SO_2(\mathbb{R})$ is a connected component of $O_2(\mathbb{R})$ and that the map

$$S^1 \to SO_2(\mathbb{R})$$
$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is an isomorphism of Lie groups.

Aufgabe 2.2: Let I_p denote the identity matrix in $Mat_p(\mathbb{R})$ and, for $p, q \in \mathbb{N}$, let

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in Mat_n(\mathbb{R})$$

where $n = p + q$.
The *indefinite orthogonal group* $O_{p,q}(\mathbb{R})$ is the set of transformation of $\mathbb{R}^n$ preserving the bilinear form defined by $I_{p,q}$, i.e.

$$O_{p,q}(\mathbb{R}) := \{A \in Mat_n(\mathbb{R}) \mid A^t I_{p,q} A = I_{p,q}\}.$$

1. Show that $O_{p,q}(\mathbb{R})$ is a closed subgroup of $GL_n(\mathbb{R})$.
2. Show that if $p, q \geq 1$ then $O_{p,q}(\mathbb{R})$ is not compact.

Hint for (2): Enough to show it for $O_{1,1}(\mathbb{R})$ (Why?).
Notice that $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in O(1,1)$, so it cannot be bounded!

Aufgabe 2.3: Show that the exponential gives an isomorphism between the vector space of upper triangular matrix with 0 on the diagonal and the group of upper triangular matrix with 1 on the diagonal. Hint: The power series of the logarithm gives a well-defined inverse.
Achtung: This is a very special case In general the exponential map is neither surjective nor injective!

Aufgabe 2.4: The goal of the exercise is to compute $T_{Id}SL_n(\mathbb{R})$.

1. Show that for $A \in GL_n(\mathbb{C})$ and $B \in Mat_n(\mathbb{C})$ we have

$$e^{ABA^{-1}} = Ae^B A^{-1}.$$

2. Show that for every $B \in Mat_n(\mathbb{C})$ we have

$$e^{Tr(B)} = \det(e^B)$$

Hint: First show it for B upper triangular, then use (1) for the general case.

3. Let $\mathfrak{sl}_n(\mathbb{R}) := \{B \in Mat_n(\mathbb{R}) \mid Tr(B) = 0\}$. Show that $\mathfrak{sl}_n(\mathbb{R}) = T_{Id}SL_n(\mathbb{R})$.
Hint: Use (2) to show that inclusion $\mathfrak{sl}_n(\mathbb{R}) \subset T_{Id}SL_n(\mathbb{R})$. The space $T_{Id}SL_n(\mathbb{R})$ cannot be larger than $\mathfrak{sl}_n(\mathbb{R})$ because...