Lie Groups
SoSe 2023 — Ubungsblatt 3

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Lie Algebras of Lie Groups

Aufgabe 3.1: Let G be a closed subgroup of $GL_n(\mathbb{R})$ and let N be a closed normal subgroup of G. Show that for any $X \in \mathrm{Lie}(G)$ and $Y \in \mathrm{Lie}(N)$ we have $[X, Y] \in \mathrm{Lie}(N)$.
(A subspace of a Lie algebra with this property is called an *ideal*.)
Hint: For any $s, t \in \mathbb{R}$ we have $e^{tX}e^{sY}e^{-tX} \in N$. Then take derivative in t and s.

Aufgabe 3.2: Let G be an abelian Lie subgroup of $GL_n(\mathbb{R})$.

- Show that Lie algebra $\mathrm{Lie}(G)$ is abelian, i.e. for every $X, Y \in \mathrm{Lie}(G)$ we have $[X, Y] = 0$.
- Regard $\mathrm{Lie}(G)$ as a group with $+$. Show that $exp : T_I G \to G$ is a group homomorphism. Moreover, if G is connected show that exp is surjective.
- Deduce that an abelian Lie subgroup of $GL_n(\mathbb{R})$ is isomorphic to $G \cong \mathbb{R}^m/\Gamma$, where Γ is a discrete subgroup of $\mathbb{R}^m$.

Aufgabe 3.3: Let V be a representation of a Lie group $G \subset GL_n(\mathbb{R})$. For $v \in V$ let $G_v = \{g \in G \mid g \cdot v = v\}$ be the stabilizer subgroup of v. Show that $\mathrm{Lie}(G_v) = \{X \in \mathrm{Lie}(X) \mid X \cdot v = 0\}$.

Connected components of the orthogonal groups

Aufgabe 3.4: Show that $SO(n, \mathbb{R}) = O_n(\mathbb{R}) \cap SL_n(\mathbb{R})$ is a connected Lie group for any $n > 1$, while $O(n, \mathbb{R})$ has two connected components.
Hint: The group $SO(n, \mathbb{R})$ acts on the sphere S^n with stabilizer isomorphic to $SO(n-1, \mathbb{R})$.