**Lie Groups**
**SoSe 2023 — Ubungsblatt 4**
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## Complexification

**Aufgabe 4.1:** Let $V^{\mathbb{C}}$ be the complexification of a real vector space $V$. Let $\overline{(-)} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ be the $\mathbb{C}$-antilinear map defined by $\overline{v+iw} = v - iw$ for all $v, w \in V$. Let $W \subset V^{\mathbb{C}}$. Show that $W$ is the complexification of a subspace of $V$ if and only if we have $W = \overline{W}$. In which case, we have $W = (W^c)^{\mathbb{C}}$, where $W^c = \{w \in W \mid \overline{w} = w\}$.

**Aufgabe 4.2:** Let $V$ be a complex representation of a group $G$. A real form $j : V \to V$ is an antilinear map (i.e. we have $j(\lambda v + \mu w) = \bar{\lambda} j(v) + \bar{\mu}(w))$ such that $j(gv) = g(jv)$ and $j^2 = \mathrm{id}$. Show that $V$ has a real form if and only if there exists a representation $W$ over $\mathbb{R}$ such that $V \cong W^{\mathbb{C}}$.

## Representation Theory of $SU(2,\mathbb{C})$ and $SO(3,\mathbb{R})$

**Aufgabe 4.3:** Let $L(n)$ be the complex irreducible representation of $SU(2,\mathbb{C})$ of dimension $n+1$. Show that $L(n)$ admits a real form if $n$ is even. Deduce that for any even positive integer $n$ there is a real irreducible representation of $M(n)$ of dimension $n+1$.

**Remark 1** *This is actually an if and only if. In fact, irreducible real representations of $SU(2,\mathbb{C})$ over $\mathbb{R}$ are the $M(n)$ for $n$ even, and $L(n)$ for $n$ odd seen as a real representation of dimension $2n+2$.*

**Aufgabe 4.4:** Consider the adjoint representation $\rho : SU(2,\mathbb{C}) \to GL(\mathfrak{su}(2,\mathbb{C}))$.

1. Show that $\langle A, B\rangle = -\mathrm{tr}(AB)$ is a $SU(2,\mathbb{C})$-invariant scalar product.
2. Deduce that there an surjective morphism of Lie groups $SU(2,\mathbb{C}) \to SO(3,\mathbb{R})$ with kernel $\{\pm Id\}$.
3. Find all irreducible real and complex irreducible representations of $SO(3,\mathbb{R})$.

**Bonus-Aufgabe 4.5:** Let $\times$ be the cross product on $\mathbb{R}^3$ (recall that if $v, w \in \mathbb{R}^3$ and $\theta$ is the angle between $v$ and $w$, then $v \times w$ is the unique vector in $\mathbb{R}^3$ of norm $\|v \times w\| = \|v\|\|w\| \sin(\theta)$, that is orthogonal to both $v$ and $w$ and such that $\det\big(v|w|v \times w\big) > 0$.

1. Show that $(\mathbb{R}^3, \times)$ is a Lie algebra isomorphic to $\mathfrak{so}_3(\mathbb{R})$.
2. Show that the Lie algebra $\mathfrak{so}_3(\mathbb{R})$ is simple and that it does not contain any Lie subalgebra of dimension 2.
3. Deduce that the group $SU_2(\mathbb{C})$ does not have any irreducible real representation of dimension 2.