Lie Groups
SoSe 2023 — Ubungsblatt 5

07.06.2023

Aufgabe 5.1: Write down a Haar measure of the Lie group $\mathbb{C}^*$.

Aufgabe 5.2: Decompose every irreducible representation of $SU(2,\mathbb{C})$ into irreducible representations of $SO(2,\mathbb{R}) \subset SU(2,\mathbb{C})$.

Aufgabe 5.3: In this exercise we give another construction of the irreducible representations of $SO(3,\mathbb{R})$. The natural action of $SO(3,\mathbb{R})$ on $\mathbb{R}^3$ induces a representation on the vector space of homogeneous polynomials $\mathbb{R}[X,Y,Z]^l$ given by $g \cdot P(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = P(g^{-1} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix})$. Call $V(l)$ this representation.

- Show that the inclusion $V(l) \hookrightarrow V(l+2)$ given by the multiplication by $(X^2+Y^2+Z^2)$ is a morphism of representations of $SO(3,\mathbb{R})$.
- Show that $V(l) \cong V(l-2) \oplus L(2l)$ where $L(2l)$ is the irreducible representation of dimension $2l+1$. Hint: Restrict to the group of rotation around the z-axis and argue by induction that $L(2l)$ is a "new" representation.

Aufgabe 5.4: Let V be an irreducible complex representation of a group G. Show that a G-hermitian scalar product on V, if it exists, is unique up to multiplication by a positive real number.

Bonus-Aufgabe 5.5: Consider the Lie group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R} \setminus 0,\ b \in \mathbb{R} \right\}.$$

Show that $\frac{1}{a^2} dadb$ is a left invariant Haar measure and $dadb$ is a right invariant Haar measure.