Lie Groups
SoSe 2023 — Ubungsblatt 8
05.07.2023

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Aufgabe 8.1: Show that

$$T_{2n} = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & & \\ & & & & \ddots & & & \\ & & & & & & \cos\theta_n & \sin\theta_n \\ & & & & & & -\sin\theta_n & \cos\theta_n \end{pmatrix} | \theta_1, \ldots, \theta_n \in \mathbb{R} \right\}$$

is a maximal torus of $SO(2n, \mathbb{R})$ and that

$$T_{2n+1} = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & & \cos\theta_n & \sin\theta_n & \\ & & & & & & -\sin\theta_n & \cos\theta_n & \\ & & & & & & & & 1 \end{pmatrix} | \theta_1, \ldots, \theta_n \in \mathbb{R} \right\}$$

is a maximal torus of $SO(2n+1, \mathbb{R})$.
Notice that both $SO(2n, \mathbb{R})$ and $SO(2n+1, \mathbb{R})$ have a maximal torus of dimension n.
Compute the Cartan subalgebras corresponding to the tori T_{2n} and T_{2n+1}.

Aufgabe 8.2: Show that a compact group of dimension 2 is a torus.
Hint: Use exercise 7.2

Aufgabe 8.3: Let $U(n, \mathbb{C})$ and $SU(n, \mathbb{C})$ be the unitary and special unitary group, i.e. the group of unitary matrices of determinant one. Compute the center of both group. Describe the adjoint compact group with the same Lie algebra of $SU(n, \mathbb{C})$. (Actually, $SU(n, \mathbb{C})$ is a simply-connected group, but we do not show it)
Let $T \subset SU(n, \mathbb{C})$ be the maximal torus of diagonal matrices. Describe the regular elements in $\mathfrak{h} = \mathrm{Lie}(T)$, i.e. the $X \in \mathfrak{h}$ such that $\mathfrak{h} = \mathrm{Ker}\, ad_X$.

Aufgabe 8.4: Let G be an abelian connected Lie group. Show that $G \cong (S^1)^r \times \mathbb{R}^s$ for some $r, s \in \mathbb{N}$.