Lie Groups
SoSe 2023 — Ubungsblatt 10

19.07.2023

Aufgabe 10.1: Consider the *compact symplectic group* $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$ where $Sp(2n, \mathbb{C}) = \{g \in GL(2n, \mathbb{C}) \mid g^t J g = J\}$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. (In other words, $Sp(n, \mathbb{C})$ is the group of transformation preserving a non-degenerate bilinear antisymmetric form.)

- Show that $\operatorname{Lie} Sp(n) = \operatorname{Lie} U(2n) \cap \operatorname{Lie} Sp(2n, \mathbb{C})$ with $\operatorname{Lie} Sp(2n, \mathbb{C}) = \{X \in Mat_{2n}(\mathbb{C}) \mid X^t J + JX = 0\}$.
- Show that $\operatorname{Lie}_{\mathbb{C}} Sp(n) \cong \operatorname{Lie} Sp(2n, \mathbb{C})$ and deduce that $\dim Sp(n) = 2n^2 + n$. Hint: $X \mapsto -\bar{X}^t$ is a real form
- Show that the diagonal matrices in $Sp(n)$ are $diag(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1})$ and form a maximal torus T of rank n.
- Let ε_i be the character of T which returns its i-th entry on the diagonal. Let $E_{i,j}$ denote an elementary matrix with 1 on the i-th column and j-th row and zero everywhere else. Verify that the following are weight space decomposition of $\mathfrak{g} := Lie_{\mathbb{C}} Sp(n)$.
  - $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}(E_{i,j} - E_{j+n,i+n})$
  - $\mathfrak{g}_{\varepsilon_i + \varepsilon_j} = \mathbb{C}(E_{i,j+n} + E_{j,i+n})$
  - $\mathfrak{g}_{-\varepsilon_i - \varepsilon_j} = \mathbb{C}(E_{i+n,j} + E_{j+n,i})$
  - $\mathfrak{g}_{2\varepsilon_i} = \mathbb{C} E_{i,i+n}$
  - $\mathfrak{g}_{-2\varepsilon_i} = \mathbb{C} E_{i+n,i}$.
- Draw the root system of $Sp(2)$ and compute its Weyl group (assuming we know it is generated by the reflections along the roots).
- **Bonus/hard**: Show that the Weyl group of $Sp(n)$ is the group of signed permutation on n elements, i.e. permutations of $\{\pm 1, \ldots, \pm n\}$ such that $\sigma(-k) = -\sigma(k)$.