Noncommutative Algebra and Symmetry
WS 2021/22 — Übungsblatt 4

08.11.2021

Exercise 4.1: Let p be a prime. Let G be a p-group (i.e. $|G| = p^n$) and let F be a field of characteristic p. Compute the Jacobson radical of FG.
(Hint: recall by exercise 1.3 that there is only a simple G-module over F)

Exercise 4.2: Write the character table over $\mathbb{C}$ of the following groups:

- $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
- the dihedral group $D_4 = \langle r, s | s^2 = r^4 = Id,\ srs = r^{-1} \rangle$.

Exercise 4.3: Let G be a group and let X be a finite set on which G acts by permutation. Let kX be a vector space with X as a basis (kX can also be thought as the vector spaces of maps $X \to k$). This is in a natural way a representation of G,
called the *permutation representation*, defined by

$$G \times kX \to kX$$

$$(g, x) \mapsto g \cdot x$$

Let χ_{kX} denote the character of kX.

1. Show that $\chi_{kX}(g)$ is equal to the number of fixed points of g in X.
2. Show that kX always contain the trivial representation as a subrepresentation.
3. Assume now that G is finite and $|G| \neq 0$ in k. Let m be the number of orbits of G in X. Show that kX contains exactly m copies of the trivial representation, i.e. we have

   $$kX \cong \overbrace{triv \oplus triv \oplus \ldots \oplus triv}^{m} \oplus \vartheta$$

   as G-representations, where ϑ is a representation of G that does not contain the trivial one.

Exercise 4.4: We are again in the setting of Exercise 4.3.3,
If the action of G on X is transitive (i.e., X has a single orbit) we have a decomposition $kX = triv \oplus \vartheta$, where ϑ does not contain the trivial representation. We give now a criteria for ϑ to be irreducible.
Assume further that the action of G on X is *doubly transitive*, that is for any x_1, x_2, y_1, y_2 with $x_1 \neq x_2$ and $y_1 \neq y_2$ there exists $g \in G$ such that $g \cdot x_1 = y_1$ and $g \cdot x_2 = y_2$.

1. Show that the action of G on $X \times X$ defined by $g \cdot (x, y) = (g \cdot x, g \cdot y)$ has exactly 2 orbits.

2. Show that $\chi_{k(X \times X)} = \chi_{kX}^2$.

3. Show that the representation ϑ is irreducible.
   Hint: we have $\chi_{kX} = 1 + \chi_\vartheta$. From $\sum_g \chi_{kX}(g)^2 = 2$ follows $\sum_g \chi_\vartheta(g)^2 = \sum_g \chi_\vartheta(g)\chi_\vartheta(g^{-1}) = 1$.