Noncommutative Algebra and Symmetry
WS 2021/22 — Ubungsblatt 8

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Exercise 8.1: Let H be a group and $G = \{1\}$. Let $f : H \to G$ be the trivial group homomorphism. (In this case f is not the inclusion!)

1. Show that $k = kG$ is isomorphic as a kH-module to kH/I, where I is the two-sided ideal
$$I = \left\{ \sum a_h h \in kH \mid \sum a_h = 0 \right\}.$$
2. Show that
$$\mathrm{coind}^1_H(V) = k \otimes_{kH} V = (kH/I) \otimes_{kH} V = V/(I \cdot V)$$
The vector space $V_H := V/(I \cdot V)$ is called the *coinvariants* of V.
3. Show that $\mathrm{ind}^1_H(V) = \mathrm{Hom}_H(k, V) = V^H$ are the H-invariants of V.
4. If H is finite then the map $b : V \to V$ defined by $v \mapsto \sum_h h \cdot v$ induces a map $b : V_H \to V^H$. Show that if $\mathrm{char}(k) = 0$, then b is an isomorphism.

Exercise 8.2: Let H be a finite group and k be a field of characteristic 0. Assume $H \subset G$ and let k be the trivial representation of H. Then
$$\mathrm{coind}^G_H(k) = kG \cdot (\sum_{h \in H} h).$$
(In particular, if Y is a Young diagram, we have $M(Y) = \mathrm{coind}^{S_Y}_C(k)$, where C is the stabilizer of the columns.)

Exercise 8.3: Show that the coinduction is transitive. If $f : G \to G'$ and $g : G' \to G''$ are group homomorphism, then
$$\mathrm{coind}^{G''}_G(V) \cong \mathrm{coind}^{G'}_G(\mathrm{coind}^{G''}_{G'}(V))$$

Exercise 8.4: Let S_{n-1} be the subgroup of elements in S_n fixing n. Using the Frobenius reciprocity show that the following statements, known as the *branching rules*, are equivalent.

1. If Y is a diagram with n boxes, then
$$res^{S_{n-1}}_{S_n}(L(Y)) = \bigoplus L(Y')$$
where the sum is over all the Young diagrams Y' obtained by removing a box from Y.
2. If Z is a diagram with $n-1$ boxes, then
$$ind^{S_n}_{S_{n-1}}(L(Z)) = \bigoplus L(Z')$$
where the sum is over all the Young diagrams Z' obtained by adding a box from Z.