Noncommutative Algebra and Symmetry
WS 2021/22 — Ubungsblatt 9

10.12.2021

Exercise 9.1: Let H be the subgroup of S_4 generated by (1234) and (13). Show that H is isomorphic to the dihedral group D_4. For any irreducible representation $V \in \mathrm{Irr}_{\mathbb{C}}(S_4)$, compute the decomposition into irreducibles of $\mathrm{res}^H_{S_4}(V)$. For any irreducible representation $W \in \mathrm{Irr}_{\mathbb{C}}(H)$, compute the decomposition into irreducibles of $\mathrm{res}^{S_4}_H(W)$.

Exercise 9.2: Let H be a subgroup of G of finite index and k a field of char 0. Let V be an irreducible representation over k of H, and let

$$\mathrm{ind}^G_H(V) \cong L_1^{a_1} \oplus \ldots \oplus L_n^{a_n}$$

be a decomposition of $\mathrm{ind}^G_H(V)$ into irreducible components (with $L_i \ncong L_j$ if $i \neq j$). Show that

$$\sum_{i=1}^n a_i^2 \leq |G/H|$$

Exercise 9.3: Let G be a finite group and N be a normal subgroup. Let k be a field of char 0. For V a representation of N and W a representation of G over k, show that

$$\mathrm{res}^H_G \, \mathrm{ind}^G_H(V) \cong \bigoplus_{g \in [G/N]} V^g$$

$$\mathrm{ind}^G_H \, \mathrm{res}^H_G(W) \cong k(G/N) \otimes_k W$$

Exercise 9.4: Let G be a finite group and N be a normal subgroup. Let k be a field of char 0. Let $V \in \mathrm{Irr}_k(G)$ and $W \in \mathrm{Irr}_k(N)$. Show that the following are equivalent:

- $\mathrm{res}^N_G(V) \cong W^a$ with $a^2 = |G/N|$
- V is the unique irreducible subrepresentation of $\mathrm{ind}(W)$ and $W^g \cong W$ for any $g \in G$.