Noncommutative Algebra and Symmetry
WS 2021/22 — Ubungsblatt 10

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Exercise 10.1: Let $G = C_p \times C_p$ and let k be a field of characteristic p. The goal of this exercise is to show that there are infinitely many isomorphism classes of indecomposable kG-modules. (Studying indecomposable representations is much harder than irreducible representations in characteristic p!)

1. Show that $kG \cong k[x,y]/(x^p, y^p)$, so a kG-module is a vector space W together with two commuting endomorphisms f, g with $f^p = g^p = 0$.
2. Let $V = V_{2n+1}$ be a k-vector space with basis $v_0, \ldots, v_n, w_1, \ldots, w_n$ and let $f(w_i) = v_{i-1}$, $f(v_i) = 0$, $g(w_i) = v_i$ and $g(v_i) = 0$. Then $f^2 = g^2 = 0$ and $fg = gf = 0$, so V is a kG-module.

The goal is to show that $V = V_{2n+1}$ is indecomposable, or equivalently that $\mathrm{End}_{kG}(V)$ is local.

3. Show that $V^G = \ker f \cap \ker g = \langle v_0, \ldots, v_n \rangle$ and $\bar{V} := V/V^G = \langle \bar{w_1}, \ldots, \bar{w_n} \rangle$, where $\bar{w_i}$ is the projection of w_i to $\bar{V}$. Moreover, f and g induce injective linear maps $\bar{V} \to V^G$.
4. Let $I = \{\psi \in \mathrm{End}_{kG}(F) \mid \mathrm{Im}(\psi) \subset V^G\rangle$. Show that I is a nilpotent ideal, hence $I \subset J(\mathrm{End}_{kG}(V))$.
5. Let $\phi \in \mathrm{End}_{kG}(V)$. Show that ϕ induces a morphism $\bar{\phi} : \bar{V} \to \bar{V}$.
6. Assume that
$$\bar{\phi}(\bar{w_1}) = \lambda_1 \bar{w_1} + \lambda_2 \bar{w_2} + \ldots + \lambda_n \bar{w_n},$$
with $\lambda_i \in k$. Use $g(w_1) = f(w_2)$ to deduce $\lambda_n = 0$ and
$$\bar{\phi}(\bar{w_2}) = \lambda_1 \bar{w_2} + \lambda_2 \bar{w_3} + \ldots + \lambda_{n-1} \bar{w_n}$$
7. Deduce that $\bar{\phi}$ is completely determined by its value in $\bar{w_1}$, and that $\bar{\phi}\bar{w_i} = \lambda_1 \bar{w_i}$ for any i.
8. Deduce that $\dim \mathrm{End}_{kG}(V)/I = 1$, hence V is local.

Exercise 10.2: Let R be a ring and M, N, E be R-modules. We say that E is an *extension* of M by N is M is a submodule of E such that $E/M \cong N$. We say that an extension E is *trivial* if $E \cong M \oplus N$.
Let k and P_ω be the only two irreducible simple representations of S_3 over k, with k algebraically closed of characteristic 2. Show that the projective cover P_1 of k is a non-trivial extension of k by k while there are no non-trivial extensions

1. of k by P_ω
2. of P_ω by P_ω
3. of P_ω by k

Exercise 10.3: Let k be an algebraic closed field of characteristic 3. Compute the dimension of the simple and of the indecomposable projective modules of kS_3.