# Noncommutative Algebra and Symmetry
# WS 2021/22 — Ubungsblatt 11

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In this exercise sheet we study *blocks* of rings.

**Exercise 11.1:** Let $R$ be a ring. We say that $e \in R$ is a *central idempotent* if it is an idempotent lying in the center $Z(R)$ of $R$.

1. Let $e \in R$ be a central idempotent. Show that $Re$ is a two-sided ideal of $R$. Moreover, $Re$ is a ring with identity $e$.
2. Show that decompositions of the $R = R_1 \oplus R_2 \oplus \ldots \oplus R_k$ into two-sided ideal are in bijection with the expressions of the form
$$1 = e_1 + \ldots + e_.$$
where the $e_i$'s are orthogonal central idempotent.
3. If $R = R_1 \oplus R_2 \oplus \ldots \oplus R_k$, we also have $R \cong R_1 \times R_2 \times \ldots \times R_k$ as rings.

**Exercise 11.2:** We say that $e \in R$ is a *primitive central idempotent* if it is a central idempotent which cannot be written as sum of two orthogonal central idempotents.

1. Show that $1 \in R$ can be written in a unique way as a sum of primitive central idempotents.
2. There exists a unique decomposition of $R = R_1 \times R_2 \times \ldots \times R_k$ into indecomposable rings.

The indecomposable rings $R_1, \ldots, R_k$ are called the *blocks* of $R$.

**Exercise 11.3:** Let $R$ be a ring and $M$ be an indecomposable $R$-module. Let $R \cong R_1 \times R_2 \times \ldots \times R_k$ be the blocks decomposition of $R$. Then $M = R_iM$ for some $i$ and $R_jM = 0$ if $j \neq i$.

**Exercise 11.4:** Let $G$ be a finite a group and $k$ a field. Then compute the blocks of $kG$ in terms of its Cartan matrix: if the Cartan matrix has a block diagonal form, then the blocks of the matrix are in bijection with the blocks of $kG$.

**Exercise 11.5:** Use the knowledge of the previously computed Cartan matrix to compute the number of blocks of $S_3$ and $A_4$ in characteristic $p = 2$ and $p = 3$.