Discrete Fourier Transform

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Let G be an abelian of order n. Let $\widehat{G} := \{\rho : G \to \mathbb{C}^* \mid \rho \text{ group homomorphism}\}$. Then $\widehat{G}$ is in bijection with the set of isomorphism classes of irreducible representations of G, which are all 1-dimensional.

Lemma 0.1. *For any $g \in G$ and $\chi \in \widehat{G}$ we have $\chi(g)^{-1} = \overline{\chi(g)}$.*

Proof. Notice that $\chi(g) \in \mathbb{C}^*$. Since g is of finite order, also $\chi(g)$ is of finite order, so $\chi(g)$ has norm 1, that is $\chi(g)\overline{\chi(g)} = 1$ and $\chi(g^{-1}) = \chi(g)^{-1} = \overline{\chi(g)}$. □

For $f : G \to \mathbb{C}$ and $\chi \in \widehat{G}$, define

$$\widehat{f}(\chi) = \frac{1}{n}\sum_{g \in G} f(g)\chi(g).$$

Proposition 0.2. *Let $f : G \to \mathbb{C}$ We have*

$$f(h) = \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\overline{\chi(h)}$$

Notice that if $\chi \in \widehat{G}$, also $\overline{\chi} \in \widehat{G}$. We are basically computing the coefficients of f when we write it in the basis $\{\chi \mid \chi \in \widehat{G}\}$.

Proof. Let $\chi_1, \chi_2, \ldots, \chi_n$ be the elements of $\widehat{G}$. We want to compute explicitly the Fourier transform for G. Artin–Wedderburn theorem for G gives an isomorphism

$$\Phi : \mathbb{C}G \xrightarrow{\sim} \mathbb{C} \times \ldots \times \mathbb{C} = \bigoplus_{\chi \in \widehat{G}} \mathbb{C} = \bigoplus_{\chi = 1}^{n} \mathbb{C}.$$

We can reinterpret any function $f : G \to \mathbb{C}$ as an element in $\mathbb{C}G$, by sending $f : G \to \mathbb{C}$ to the element $\sum_g f(g)g \in \mathbb{C}G$.

Let $g \in G$ and think of it as an element of $\mathbb{C}G$. Then

$$\Phi(g) = (\chi_1(g), \chi_2(g), \ldots, \chi_n(g))$$

In the other direction, let

$$e_i := (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \bigoplus_{\chi = 1}^{n} \mathbb{C},$$

where the 1 is in the ith position. The computation of the inverse Fourier transform tells us that

$$\Phi^{-1}(e_i) = \frac{1}{n}\sum_{g \in G} \chi_i(g^{-1})g = \frac{1}{n}\sum_{g \in G} \overline{\chi_i(g)}g$$

In other words, $\Phi^{-1}(e_i) = \frac{1}{n}\overline{\chi_i}$ as functions. Let now $f : G \to \mathbb{C}$ and interpret it as the element $\sum_g f(g)g \in \mathbb{C}G$. We have

$$\begin{aligned}
f = \Phi^{-1}\Phi(f) &= \Phi^{-1}\left((\sum_{g\in G} f(g)\chi_1(g), \sum_{g\in G} f(g)\chi_2(g), \dots, \sum_{g\in G} f(g)\chi_n(g)) \right) \\
&= \Phi^{-1}\left((n\widehat{f}(\chi_1), n\widehat{f}(\chi_2), \dots, n\widehat{f}(\chi_n)) \right) \\
&= n \cdot \Phi^{-1}\left(\sum_{i=1}^{n} \widehat{f}(\chi_i)e_i \right) \\
&= n\sum_{i=1}^{n} \widehat{f}(\chi_i)\Phi^{-1}(e_i) = n\sum_{i=1}^{n} \widehat{f}(\chi_i)\frac{1}{n}\overline{\chi_i} = \sum_{i=1}^{n} \widehat{f}(\chi_i)\overline{\chi_i}
\end{aligned}$$

In other words, this tells us that for any $h \in G$ we have

$$f(h) = \sum_{i=1}^{n} \widehat{f}(g)\overline{\chi_i(h)} = \sum_{\chi\in\widehat{G}} \widehat{f}(g)\overline{\chi(h)}.$$

□

Example 0.3. Let $G = C_n$ be the cyclic group of n elements. We can think of G as $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, that is as the group with elements $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}$. Then $\widehat{G} = \{\chi_1, \dots, \chi_n\}$ where

$$\begin{aligned}
\chi_j &: C_n \to \mathbb{C}^* \\
\frac{k}{n} &\mapsto e^{\frac{2\pi ikj}{n}},
\end{aligned}$$

i.e., $\chi_j(z) = e^{2\pi izj}$. Let $f : C_n \to \mathbb{C}$. We have

$$f(z) = \sum_{j=1}^{n} \widehat{f}(\chi_j)\overline{\chi_j(z)} = \sum_{j=1}^{n} \widehat{f}(\chi_j)e^{-2\pi izj}$$

where

$$\widehat{f}(\chi_j) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) e^{\frac{2\pi ijk}{n}}$$

Compare this with the usual expansion of a function in a Fourier series. The discrete Fourier series is the approximation on n points. If $f : [0, 1] \to \mathbb{C}$ is L^2-integrable then

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(j)e^{2\pi ikx}$$

where

$$\widehat{f}(j) = \int_0^1 f(x)e^{-2\pi ixj}$$