# The Hard Lefschetz Theorem for the Flag Variety in Positive Characteristic

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## The Flag Variety of a Reductive Group

Let $G$ be a simply connected reductive complex linear algebraic group, $T$ be a maximal torus and $B \supseteq T$ be a Borel subgroup.
The homogeneous space $X = G/B$ is called the **flag variety** of the group $G$. Let $W = N_G(T)/T$ be the Weyl group of $G$ with length function $\ell$ and let $\Phi$ be the root system of $G$.
The dimension $d$ of $X$ is equal to the number of positive roots $|\Phi^+| = |\Phi|/2$.

**Example.** *The fundamental example to keep in mind is the following. Let $G = SL_n(\mathbb{C})$. The subgroup of diagonal matrices $T$ is a maximal torus and the subgroup $B$ consisting of upper triangular matrices is a Borel subgroup of $G$ containing $T$. The flag variety can be identified with the set of flags in $\mathbb{C}^n$:*

$$\mathit{Flag}(\mathbb{C}^n) = \{(V_i)_{0\leq i\leq n} \mid \dim V_i = i \text{ and } V_i \subseteq V_{i+1} \text{ for all } i\}$$

*In fact, $G$ acts transitively on the set $\mathit{Flag}(\mathbb{C}^n)$ and the stabilizer of the "standard" flag*

$$0 \subseteq \mathbb{C}e_1 \subseteq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subseteq \ldots \subseteq \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n = \mathbb{C}^n$$

*is exactly the group of upper triangular matrices $B$.*
*The Weyl group $W$ is isomorphic to the symmetric group $\mathcal{S}_n$ and, for $w \in \mathcal{S}_n$,*

$$\ell(w) = \#\{(i,j) \mid 1 \leq i < j \leq n \text{ and } w(i) > w(j)\}.$$

*The set of positive roots is $\Phi^+ = \{\varepsilon_i - \varepsilon_j, 1 \leq i < j \leq n\}$ where $\varepsilon_i : T \to \mathbb{C}^*$ is the map returning the $i$-th entry on the diagonal. In particular $|\Phi^+| = \dfrac{n(n-1)}{2}$.*

We recall the following:

**Theorem** (Bruhat Decomposition). *The $B$-action on $X$ given by left multiplication decomposes the flag variety in a finite number of orbits, each of which is of the form $B\dot{w}B/B$, where $\dot{w} \in G$ is a representative for $w \in W$. Every orbit $B\dot{w}B/B$ is isomorphic, as a variety, to $\mathbb{C}^{\ell(w)}$. The closure $X_w = \overline{B\dot{w}B/B}$ is called a* Schubert variety *and is a union of $B$-orbits. More precisely,*

$$X_w = \bigsqcup_{v\leq w} B\dot{v}B/B$$

*where $\leq$ is the Bruhat order.*

The set of fundamental classes $[X_w] \in H_{2\ell(w)}(X,\mathbb{Z})$ of Schubert varieties $X_w$ is a basis of the integral homology $H^*(X,\mathbb{Z})$ of $X$.
By taking the dual, we obtain a basis $\{P_w\}_{w\in W}$, with $P_w \in H^{2\ell(w)}(X,\mathbb{Z})$, of the integral cohomology $H^*(X,\mathbb{Z})$.
We call $\{P_w\}_{w\in W}$ the **Schubert basis**.
Since $G$ is simply connected the group $H^2(X,\mathbb{Z})$ can be identified with the group of characters

$$X^*(T) = \{\phi : T \to \mathbb{C}^* \text{ hom. of algebraic groups}\}.$$

We recall the **Chevalley-Pieri's formula**. Let $\lambda \in H^2(X,\mathbb{Z})$ a weight. Then

$$\lambda \cdot P_w = \sum_{w\xrightarrow{\gamma}v} \langle \lambda, \gamma^\vee\rangle P_v.$$

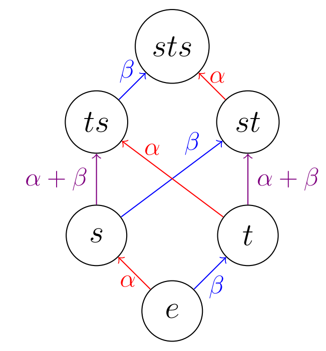
Here $w \xrightarrow{\gamma} v$, with $w, v \in W$ and $\gamma \in \Phi^+$, means that $wt_\gamma = v$ and $\ell(w)+1 = \ell(v)$, where $t_\gamma$ is the reflection corresponding to $\gamma$. The pairing $\langle\ ,\ \rangle$ is the usual pairing between weights and coroots and $\gamma^\vee$ denotes the coroot corresponding to $\gamma$ in the dual root system.

## The Bruhat Graph

We can visualize the Chevalley-Pieri's formula using a graph.
Let $\Phi$ be the root system of $G$ and $W$ be its Weyl group. We consider the **Bruhat graph** $\mathfrak{B}$ of $\Phi$:

- The vertices of the graph are the element of $W$;
- for any $v, w \in W$ such that $w \xrightarrow{\gamma} v$ there is an arrow labeled by $\gamma$ in the graph.

**Example.** *If $G = SL_3(\mathbb{C})$, then $\Phi$ is the root system of type $A_2$ and $W \cong \mathcal{S}_3$.*
*The group $\mathcal{S}_3$ is generated by the simple transposition $s = (12)$ and $t = (23)$. Let $\alpha$ and $\beta$ be the simple roots corresponding to $s$ and $t$. Then the Bruhat graph $\mathfrak{B}$ of $\Phi$ is the following:*




## The Hard Lefschetz Theorem in Characteristic 0

**Theorem** (Hard Lefschetz Theorem for the Flag Variety). *Let $\lambda$ be a dominant regular weight, i.e. $(\lambda,\alpha) > 0$ for any positive root $\alpha$. Then, for any $0 \leq k \leq n$, $\lambda^k : H^{d-k}(X,\mathbb{Q}) \to H^{d+k}(X,\mathbb{Q})$ is an isomorphism.*

The original proof of the Hard Lefschetz Theorem is geometric and holds for any smooth projective variety $X$ (and also for any singular variety, after replacing the cohomology $H^*(X,\mathbb{Q})$ with the intersection cohomology $IH^*(X,\mathbb{Q})$).

**Remark.** *In general the map $\lambda^k : H^{d-k}(X,\mathbb{Z}) \to H^{d+k}(X,\mathbb{Z})$ is not an isomorphism for $\lambda$ dominant regular. As in the example above let $G = SL_3(\mathbb{C})$. Let $\varpi_\alpha, \varpi_\beta \in H^2(X,\mathbb{Z})$ be the fundamental weights (i.e. $\langle\varpi_\alpha,\alpha\rangle = 1$ and $\langle\varpi_\alpha,\beta\rangle = 0$ and similarly for $\varpi_\beta$).*
*Then any $\lambda \in H^2(X,\mathbb{Z})$ can be written as $\lambda = a\varpi_\alpha + b\varpi_\beta$, with $a, b \in \mathbb{Z}$. We have:*

- $\det(\lambda : H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z})) = a^2 + ab + b^2$ *is invertible if and only if*
$$(a,b) \in \{(\pm1,0),(0,\pm1),(1,-1),(-1,1)\};$$
- $\det(\lambda^3 : H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z})) = 3ab(a+b)$ *is never invertible.*

*However, if $\mathbb{K}$ is a field of characteristic $> 3$, then $\lambda = \varpi_\alpha + \varpi_\beta$ satisfies Lefschetz on $H^*(X,\mathbb{K}) = H^*(X,\mathbb{Z}) \otimes \mathbb{K}$.*

We say that $\lambda$ *satisfies Hard Lefschetz* on $H^*(X,\mathbb{K})$ if $\lambda^k : H^{d-k}(X,\mathbb{K}) \to H^{d+k}(X,\mathbb{K})$ is an isomorphism for every $0 \leq k \leq d$.

## The Hard Lefschetz Theorem in Positive Characteristic

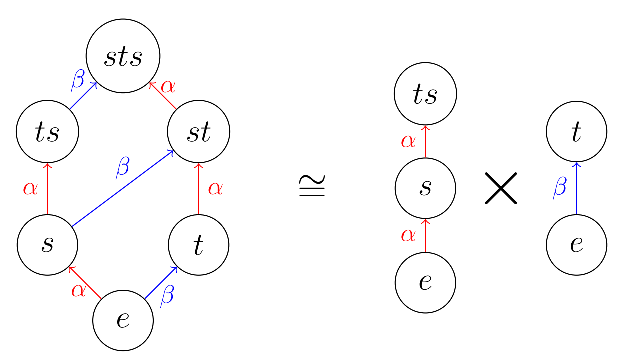
**Theorem 1.** *Let $\mathbb{K}$ an infinite field of characteristic $p > 0$. Then if $p > |\Phi^+|$ there exists $\lambda \in H^2(X,\mathbb{K})$ satisfying Hard Lefschetz on $H^*(X,\mathbb{K})$.*

*Sketch of the proof.* We denote by $\{1,2,\ldots,n\}$ the simple roots in $W$. Let $W_i$ the subgroup of $W$ generated by the simple reflection $s_i, s_{i+1},\ldots,s_n$.
Let $\lambda = \sum_{i=1}^n x_i\varpi_i$ a generic element of $H^2(X,\mathbb{K})$, with $\varpi_i$ fundamental weights and $x_i \in \mathbb{K}$. The $x_i$ can be thought as variables and the determinant of $\lambda^k : H^{n-k}(X,\mathbb{C}) \to H^{n+k}(X,\mathbb{C})$ is then a polynomial $D_k(x_1,\ldots,x_n)$.
Since $\mathbb{K}$ is infinite, it is enough to show that all the polynomials $D_k$, $0 \leq k \leq n$ are not identically zero.
We define the degenerate Bruhat Graph $\mathfrak{B}^{\deg}$ as follows:

- the vertices of $\mathfrak{B}^{\deg}$ are the same of $\mathfrak{B}$;
- we can write any $w$ in a unique way as $w^{(1)}w^{(2)}\ldots w^{(n)}$ where $w^{(i)} \in W_i$ has minimal length in its class in $W_i/W_{i+1}$. Then the arrows in $\mathfrak{B}^{\deg}$ are the arrows $w \xrightarrow{\gamma} v$ in $\mathfrak{B}$ such that $v^{(i)} \geq w^{(i)}$ for any $i$.

  The label of the new arrows is obtained replacing the label $\gamma$ by its leading term in the lexicographic order.

**Example.** *The Degenerate Bruhat Graph $\mathfrak{B}^{\deg}$ for $G = SL_3(\mathbb{C})$:*




The new graph $\mathfrak{B}^{\deg}$ describes a new action of $\lambda \in H^2(X,\mathbb{K})$ on $H^*(X,\mathbb{K})$ by $\lambda \cdot P_w = \sum_{w\xrightarrow{\gamma}v\in\mathfrak{B}^{\deg}} \langle\lambda,\gamma^\vee\rangle P_v$.
If we compute the determinant of $\lambda^k$ with respect of this new action we obtain the leading term (in the lexicographic order) of $D_k$. Thus it remains to show that this new action satisfies the Hard Lefschetz Theorem.
Let's assume $G = SL_{n+1}(\mathbb{C})$ (the other cases are analogous, after some work). Then $\mathfrak{B}^{\deg}$ is isomorphic to a product of strings. Now the theorem is an easy consequence of the following result. $\square$

**Theorem** ([Pro90], [Coo12]). *The ring $A = \mathbb{K}[a_1,\ldots,a_n]/(a_1^{d_1},\ldots,a_n^{d_n})$ is a finite dimensional graded algebra, where $\deg a_i = 2$. Then if $\mathit{char}(\mathbb{K}) > \sum(d_i - 1)$ there exists $\lambda \in A$ of degree 2 satisfying Hard Lefschetz on $A$.*

**Remark.** *Conversely, if there exists $\lambda \in H^2(X,\mathbb{Z})$ satisfying the Hard Lefschetz Theorem on $H^*(X,\mathbb{K})$ then $\mathit{char}(\mathbb{K}) > |\Phi^+|$ unless we are in one of the following cases:*

- *$X$ of type $A_2$ (e.g. $G = SL_3(\mathbb{C})$), hence $|\Phi^+| = 3$, and $\mathit{char}(\mathbb{K}) = 2$;*
- *$X$ of type $B_2$ (e.g. $G = SO_5(\mathbb{C}) = Sp_4(\mathbb{C})$), hence $|\Phi^+| = 4$, and $\mathit{char}(\mathbb{K}) = 3$;*
- *$X$ of type $G_2$, hence $|\Phi^+| = 6$, and $\mathit{char}(\mathbb{K}) = 5$.*

## Motivations in Representation Theory: Lusztig's Conjecture

Let $G$ a connected reductive algebraic group over an algebraic closed field $\mathbb{K}$ of characteristic $p$.
The **Lusztig's conjecture** [Lus80] is a formula that allows to compute the characters of irreducible representations of $G$ over $\mathbb{K}$ in terms of the affine Kazhdan-Lusztig polynomials.
Lusztig's conjecture is proven [AJS94] for $p \gg n$ =rank($G$), but the only explicit bounds known are huge [Fie12] (at least $p > n^{n^2}$).
On the other hand Williamson [Wil13] found a family of counterexamples to the conjecture with $p = O(c^n)$ and $c = 1,101\ldots$.
We still do not know, between these two bounds, when Lusztigs conjecture starts to hold!
Via geometric approaches developed by Soergel, Fiebig, Riche-Williamson it seems likely that the hard Lefschetz theorem controls Lusztig's conjecture to some extent, i.e. if the hard Lefschetz holds for the intersection cohomology of certain Schubert varieties, then Lusztig's conjecture holds.
Thus, Theorem 1 can be thought as the very first step in this direction: investigate Hard Lefschetz in positive characteristic to refine the range where Lusztig's conjecture holds.

## References

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