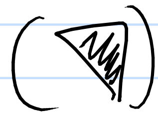
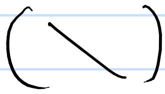
G reductive group / $\mathbb{C}$ e.g. $SL_n(\mathbb{C})$

$\cup\!\!\mid$ $\cup\!\!\mid$

B Borel subgroup 

$\cup\!\!\mid$ $\cup\!\!\mid$

T maximal torus 

W Weyl group $W = S_{n-1}$

$X = G/B$ flag variety

<u>Bruhat decomposition</u> $B = \coprod BwB/B$

$X_w = \overline{BwB}/B$ Schubert variety.

<u>Slogan</u> The geometry of Schubert variety is a very rich source of <u>information in representation</u> theory.

A fundamental object to consider is:

$IC(X_w, \mathbb{R})$ intersection cohomology sheaf of X_w

$\cap$

$D^b_c(X_w)$ derived category of constructible sheaves (wrt stratification of B-orbit)

$IH(X_w, \mathbb{R}) = H(IC(X_w, \mathbb{R}))$ intersection cohomology.

IH is the correct space to consider if we want generalize several properties of the cohomology of smooth variety:

•) IH satisfies the Poincaré duality.

$$IH^k(X_w,\mathbb{R}) \cong \left[IH^{-k}(X_w,\mathbb{R})\right]^*$$

•) IH has a pure Hodge structure

•) Hard Lefschetz theorem.

$\lambda = c_1(\mathcal{L})$ first Chern class of an ample line bundle

Then $\cdot\lambda^k : IH^{-k}(X_w,\mathbb{R}) \xrightarrow{\sim} IH^k(X_w,\mathbb{R})$

is iso $\forall\, k \geq 0$

•) Hodge Riemann Bilinear Relations

$\langle\ ,\ \rangle : IH^{-k}\times IH^{k} \longrightarrow \mathbb{R}$ intersection pairing

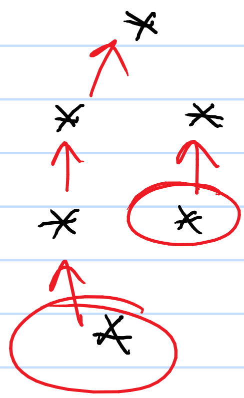
$(a,b)_\lambda := \langle a, \lambda^k b\rangle$ symmetric form on IH^{-k}

Then HLBT gives a formula to compute the signature on $(\ ,\ )_\lambda$ (in particular it is definite on $\ker \lambda^{k+1}$)

<u>EXAMPLE</u> $G = SL_3(\mathbb{C})$ $X = G/B$

X smooth

$\Rightarrow H^*(X,\mathbb{R})[3] = IH^*(X,\mathbb{R})$



$\lambda = c_1(\mathcal{L})$ $\mathcal{L}$ ample on X projective variety.

$h : IH^\cdot(X, \mathbb{R}) \to IH^\cdot(X, \mathbb{R})$ map which is multiplication by k in degree k.

HLT $\Rightarrow$ There exists $f : IH^\cdot(X, \mathbb{R}) \to IH^{\cdot-2}(X, \mathbb{R})$

such that $\{f, h, \cdot\lambda\}$ is a Lie subalgebra $\subseteq \operatorname{End}(IH^\cdot(X, \mathbb{R}))$

isomorphic to $\mathfrak{sl}_2(\mathbb{R})$

(Decomposing IH^* wrt this $\mathfrak{sl}_2(\mathbb{R})$ into irreducible components give rises to the primitive decomposition of IH^*)

What happens if we consider all these $\mathfrak{sl}_2$-triples at once?

<u>Def</u> (Looijenga-Lunts, '97) X projective variety

$\mathfrak{g}_{NS}(X)$ is the Lie subalgebra of $\operatorname{End}(IH^*(X, \mathbb{R}))$ generated by all the $\mathfrak{sl}_2$-triple $\{f, h, \cdot\lambda\}$ with $\cdot\lambda$ ample

<u>Rmk</u> • The name comes from the Neron-Severi group $NS(X)$, which is the space generated by all ample classes of line bundles on X.

• Even if multiplication by λ and λ' always commute,

the respective f's need not commute
Hence $\mathfrak{g}_{NS}(X)$ can be quite large in general.

<u>Thm</u> (LL) $\mathfrak{g}_{NS}(X)$ is a semisimple Lie algebra

This is a direct consequence of the HR Bil. Rel.

(Now I want to tell you how already this first result has an application for Kazhdan-Lusztig polynomials.)

Let's go back to Schubert varieties.

$$X^* = \{\chi : T \to \mathbb{C}^*\} = \{\chi : B \to \mathbb{C}^*\}$$

For any χ can define a line bundle $\mathcal{L}_\chi$ on G/B

by $\quad G \times_B \mathbb{C}_\chi \longrightarrow G/B$

$\chi \longmapsto c_1(\mathcal{L}_\chi)$ defines a map

$X^* \to H^2(X_w, \;)$ that can be extended to

a map $\quad R = \mathrm{Sym}(\underset{\substack{\| \\ X^* \otimes_{\mathbb{Z}} \mathbb{R}}}{\mathfrak{h}^*}) \longrightarrow H^\bullet(X_w, \mathbb{R})$

This defines a structure of R-module on $IH^*(X_w)$

<u>ERWEITERUNGSSATZ</u> (Soergel '90)

$\Rightarrow \quad IH^\bullet(X_w, \mathbb{R})$ is indecomposable as a R-module

$\mathfrak{g}_{NS}(X_w)$ is semisimple $\Rightarrow$ $IH(X_w)$ is irreducible as a $\mathfrak{g}_{NS}(X_w)$-module

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$H^\cdot(X_w, \mathbb{R}) \subseteq IH^\cdot(X_w, \mathbb{R})$ is a R-submodule

TFAE

The Betti numbers of $H^*(X_w)$ are symmetric $\iff$ $\{v \in W \mid v \leq w, \ell(v) = k\}$ $\overset{\shortparallel}{=}$ $\{v \in W \mid v \leq w, \ell(v) = \ell(w) - k\}$

$(\dim H^k = \dim H^{\ell(w)-k} \ \forall k)$

$\Updownarrow$

$H^*(X_w)$ is a $\mathfrak{g}_{NS}(X_w, \mathbb{R})$-submod. of IH^*

$\Updownarrow$

$H^*(X_w) = IH^*(X_w)$

$\Updownarrow$

All the Kazhdan-Lusztig polynomials $h_{x,w}(v)$, $x \leq w$ are trivial

(i.e. $h_{x,w}(v) = v^{\ell(x)-\ell(w)}$)

This was (of course) already known (was proved for example by Carrell-Peterson)

Here we find an easy Hodge-theoretic proof of this...

EXCURSUS: How can generalize this to a general Coxeter group?

W Coxeter group, $\mathfrak{h}^*$ geom. representation

$IH(X_w) \longleftrightarrow \overline{B_w} = B_w \otimes_R \mathbb{R}$

indecomposable Soergel bimodules

is a module over $R = \mathrm{Sym}(\mathfrak{h}^*)$

<u>Thm</u> (E-W) For $\rho \in \mathfrak{h}^*$ "ample"

$\overline{B_w}$ satisfies HLT and HR Bil rel.

We can also find an algebraic replacement homology submodule H_w.

<u>Problem</u> In general $\overline{B_w}$ is not indecomposable as a R-module (<u>EXAMPLE</u> $w = tstuts$ in $\tilde{A}_2$)

<u>To get around this</u>

Can define Z algebraic replacement of $H_T^*(X)$ (aka dual nil Hecke ring of Kostant and Kumar)

<u>Thm</u> (Fiebig) The category of Soergel bimodules is equivalent to <u>a category of Z-modules.</u>

$\overline{Z} = Z \otimes_R \mathbb{R}$, $\overline{B_w}$ is indecomposable as $\overline{Z}$-modules

$\Rightarrow$ the same result follows...

It is also natural to determine what is $\mathfrak{g}_{NS}(X_w)$ for a Schubert variety X_w.

$$\varphi(a,b) = (-1)^{\frac{k(k-1)}{2}} \langle \ , \ \rangle$$

intersection form.

φ is a non-degenerate symmetric or antisymmetric form (depending on $\dim X_w$)

$$\mathrm{aut}(\varphi) = \{ f : H^* \to H^* \mid \varphi(f(a),b) + \varphi(a, f(b)) = 0 \ \forall a,b \}$$

$$\cup\!|$$

$$\mathfrak{g}_{NS}(X_w)$$

Thm (LL) If X is a flag variety, then

$$\mathfrak{g}_{NS}(X) = \mathrm{aut}(\varphi).$$

$w \in W$, We define a directed graph $\tilde{I}_w$

- vertices : $s \in S$, $s \leq w$
- arrows : $s \to t$ if $ts \leq w$ and $ts \neq st$

A_3 $s - t - u$ then $w = utus$

$$I_w \qquad s \to t \rightleftarrows u$$

Thm (P.) If $\tilde{I}_w$ is a connected graph with no sinks

then $\mathfrak{g}_{NS}(X_w) = \mathrm{aut}(\varphi)$.

Some example In general it is not a necessary condition.

What the Hodge theory can say in positive characteristic?

Here there is no Hodge theory...
HRBR do not make sense.

Local Hard Lefschetz holds for Schubert variety controls part of the Lusztig conjecture on the character of simple modules for reductive groups $/\mathbb{k}$

$(e.g.\ G = SL_n(\mathbb{F}_p))$

I started by looking at the first non-trivial example

<u>Thm</u> X flag variety of G reductive group.
$\mathbb{k} = \overline{\mathbb{k}}$, char $\mathbb{k} = p$.
Then if $p > \dim X$ there exists $\lambda \in H^2(X, \mathbb{k})$ such that λ satisfy HL on $H^*(X)$

The hope is the one can also explicitly compute when Local HL holds for (affine) Schubert variety and this could bring to new bounds in Lusztig's conjecture.