WHAT IS HL THEOREM? LET'S QUICKLY RECALL IT

X smooth proj. alg. variety $/\mathbb{C}$ of dim d. $\rho \in H^2(X,\mathbb{C})$ $\rho = c_1(L)$ with L ample line bundle

$\Rightarrow \cdot\rho^k : H^{d-k}(X,\mathbb{C}) \to H^{d+k}(X,\mathbb{C}) \quad \forall\, 0 \le k \le d$

If X singular need to use $IH^*(X,\mathbb{C})$ in place of $H^*(X,\mathbb{C})$

Let now $G \supseteq B \supseteq T$ (e.g. $G = SL_n(\mathbb{C})$)

semisimple alg. group $/\mathbb{C}$ — Borel — Max. Torus

$W = N_G(T)/T$ Weyl group

$X = G/B$ flag variety. $X_w = \overline{BwB}/B$ Schubert variety.

$\mathfrak{g} = \mathrm{Lie}(\mathfrak{g})$. $\Delta(\lambda)$ Verma modules

$L(\lambda)$ Simple modules Q What is $\mathrm{ch}\, L(\lambda)$?

KL conj — KL poly: can be defined combinatorially

$$\mathrm{ch}\, L(w\cdot 0) = \sum (-1)^{\ell(v)-\ell(w)} h_{v,w}(q)\, \mathrm{ch}\, \Delta(v\cdot 0)$$

1st Geometric proof (Beilinson-Bernstein, Brylinski-Kashiwara)

Idea: KL conj can be translated in a question about perverse sheaves on the flag varieties

In particular $h_{v,w}(q) = \mathrm{grdim}\; i_v^* IC(X_w, \mathbb{C})$

ALGEBRAIC PROOF OF KL CONJ (Elias-Williamson, 2012 based on work by Soergel)

Here the importance of HL is more explicit

Strategy: ·) define $IH^*(X_w,\mathbb{C})$ algebraically $\rightsquigarrow B_w$ Soergel bimodules

⋮

·) Show HL for B_w ← CRUCIAL STEP

⋮

(·) Interpret B_w directly in terms of category $\mathcal{O}$ (via $V = \mathrm{Hom}(P(w_0\cdot 0), -)$))

⋮

Let's go to the char p world

Let k be a field of char p.

In general $IH^*(X_w, k)$ does not satisfy HL (for any weight λ)
But...

$\exists \lambda$ satisfying HL on every $IH^*(X_w,k)$ $\forall w$

$\Rightarrow$ the p-KL polynomials are equal to the usual KL polynomials

$\Rightarrow$ Consequences One part of Lusztig's conjecture is true ("around the Steinberg weight")

STATUS OF LUSZTIG'S CONJECTURE

Let G a semi simple algebraic group $/\overline{\mathbb{F}}_p$

Lusztig's conjecture is a formula for the character of simple modular representation

(INSERT FORMULA
$$L(x\cdot_p \mu) = \sum_{\substack{y \leq x \\ y\cdot_p \mu \in X^+}} \varepsilon_{yx} h_{y,x}(1) \Delta(x\cdot_p \mu)$$
)

- (Andersen-Jantzen-Soergel '94) It is true for $p \gg h$ ← Coxeter number of G

(Fiebig '12) gave an explicit bound (e.g. $p > 2^{2^n}$ for $SL_n(\mathbb{C})$)

- (Williamson '13) provided a family of counterexamples for $p \sim c^h$ ($c \sim 1,10\ldots$)

Moral By investigating when HL holds in characteristic p one could hope to obtain new bounds for Lusztig's conj.

This will be our dream. We start considering the first interesting example: When does HL holds on $H^*(X,k)$?

BASIC FACTS ABOUT $H^*(X,k)$

X has a cell dec. $X = \bigcup B\cdot wB \cong \mathbb{C}^{\ell(w)}$

of cell of even real dimension $H_*(X,k) = \bigoplus k[X_w]$

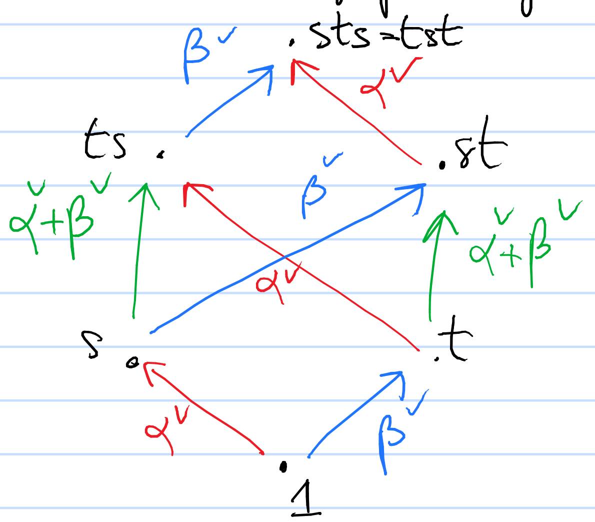
$\Longrightarrow$ $H^*(X,k) \simeq H^*(X,\mathbb{Z})\otimes k \cong \bigoplus_{w\in W} k P_w$ ← dual basis.

<u>Multiplication on $H^*(X,k)$</u> "Pieri's rule"

$$\lambda \cdot P_w = \sum_{\substack{t_\gamma \text{ reflection} \\ \ell(wt_\gamma)=\ell(w)+1}} \langle \lambda, \gamma^\vee \rangle P_{wt_\gamma}$$

We can encode this in a graph: e.g. $G = SL_3$

$W \cong S_3 = \langle s,t \mid s^2 = t^2 = (st)^3 = 1 \rangle$



Now, a general λ can be written as $\sum x_i P_{s_i} = \sum x_i \omega_i$, $x_i \in \mathbb{F}_p$

fundamental weights

$d = |\Phi^+| = \dim X$

$\lambda^k : H^{d-k}(X,k) \longrightarrow H^{d+k}(X,k)$ is iso for some λ

$\det(\lambda^k) = D_k(x_1, \dots, x_n)$ is a non-zero polynomial.

Let's do the case $k=d$. $\lambda^d \cdot P_e = c P_{w_0}$. What is c?

Thm (Stembridge 2002)

$$\lambda^d \cdot P_e = |\Phi^+|! \prod \frac{\langle \lambda, \alpha^\vee \rangle}{ht(\alpha)} P_{w_0}$$

Cor No hope for HL if $\displaystyle p \mid \frac{|\Phi^+|}{\prod ht(\alpha)}$

(in all but six cases, namely $\underbrace{A_2, B_2, G_2, B_3, C_3, F_4}_{5}$,

this is equivalent to $p \leq |\Phi^+|$

Note however that we do have λ s.t. HL on A_2 w/ $p=2$

B_2 w/ $p=3$

G_2 w/ $p=5$

Unfortunately this seems hard to compute...

Let's choose a lexicographic order $x_1 > x_2 > \dots > x_n$

Goal If we show that the leading term of $D_u(x_1, \dots, x_n)$ in the lexicographic order is non zero, we are done.

We will sketch the proof for simplicity in type A_n, i.e. $G = SL_{n+1}(\mathbb{C})$

DEGENERATION OF THE BRUHAT GRAPH

$W \cong S_{n+1}$ Weyl group $\quad s_1, \dots, s_n$ simple reflection.

$$\begin{array}{ccccccc} I_0 & \supseteq & I_1 & \supseteq & I_2 & \supseteq \dots \supseteq & I_{n-1} \\ \parallel & & \parallel & & \parallel & & \parallel \\ \{s_1, \dots, s_n\} & & \{s_2, \dots, s_n\} & & \{s_3, \dots, s_n\} & & \{s_n\} \end{array}$$

$W_i = \langle I_i \rangle$ subgroup

$W_i / W_{i+1} \longleftrightarrow$ min. representatives in W_i

$$W \overset{1:1}{\longleftrightarrow} W/W_1 \times W_1/W_2 \times \dots \times W_{n-1}$$

$$w \longmapsto (w^{(1)}, w^{(2)}, \dots, w^{(n-1)})$$

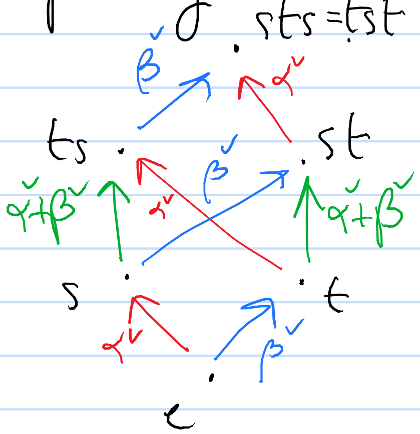
•) We keep only the edge $w \to v$ s.t. $v^{(i)} \geq w^{(i)} \ \forall i$

•) We replace a coroot with its "leading term"

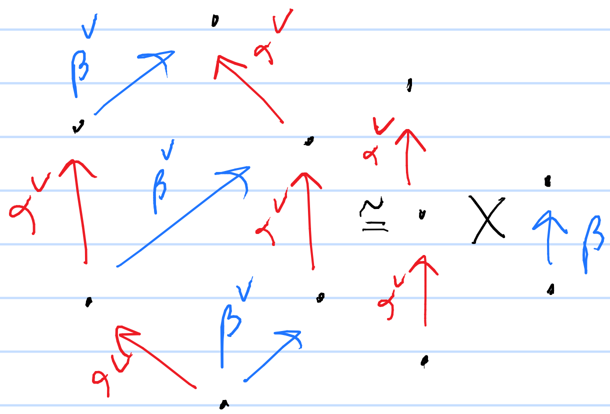
EXAMPLE $G = SL_3$ $\quad S = \{s,t\}$ $\quad s > t$.

Graphically $\qquad \alpha > \beta$

$ts = ts \cdot e$, $\quad e \in W/\langle t\rangle$



$\rightsquigarrow$



$t \longrightarrow ts$ is not allowed $\quad ts = ts \cdot 1 \quad 1 \neq t$.

$t = 1 \cdot t$

Rmk Another way to look at this degeneration

The new graph describes an action of

$$\lambda \circlearrowright H^*(G/P_1 \times P_1/P_2 \times \dots \times P_{n-1}/B, k) = H^*(G/P_1, k) \otimes H^*(P_1/P_2, k) \otimes \dots$$

$G \supseteq P_1 \supset P_2 \supset \dots \supseteq P_{n-1} \supset B$ are parabolic subgroup

corr. to $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots \supseteq I_{n-1} \supseteq \emptyset$

and λ acts as

$$x_1\omega_1 \otimes 1 \otimes 1 \dots \otimes 1 + 1 \otimes x_2\omega_2 \otimes 1 \otimes \dots 1 + \dots \quad 1 \otimes 1 \otimes \dots 1 \otimes x_n w_n$$

on $H^*(G/P_1, k) \otimes H^*(P_1/P_2, k) \otimes \dots \otimes H^*(P_{n-1}/P_n, k)$

Note that $\quad P_i/P_{i+1} \cong \mathbb{P}^{n-i}(\mathbb{C})$

The new action of λ corresponds to the action of

$$\circledast \quad \sum x_i \varpi_i \curvearrowright k[\varpi_1,\dots,\varpi_n]/(\varpi_1^{n+1}, \varpi_2^{n}, \dots, \varpi_n^{2})$$

<u>Lemma</u> $\circledast$ satisfies HL if $p > 1+2+\dots+n = |\Phi^+|$

<u>Idea</u> $x_i\varpi_i \curvearrowright k[\varpi_i]/(\varpi_i^{k})$ can be extended to a $\mathfrak{sl}_2(k)$ rep. where $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ acts as $x_i\varpi_i$

$k[\varpi_i]/(\varpi_i^{k}) \cong L(k-1)$.

If $\sum d_i < p$ then $\bigotimes L(d_i)$ is semisimple

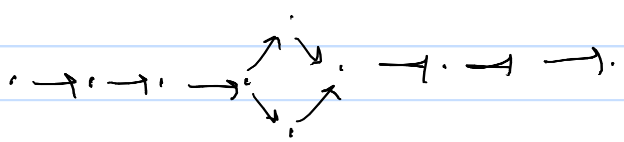
$\Rightarrow \begin{pmatrix}0&1\\0&0\end{pmatrix} \curvearrowright \bigotimes L(d_i) \cong k[\varpi_1,\dots,\varpi_n]/(\varpi_1^{d_1},\dots,\varpi_n^{d_n})$ satisfies HL.

<u>Reassuming</u> HL for $k[\varpi_1,\dots,\varpi_n]/(\varpi_1^{n+1},\dots,\varpi_n^{2})$

$\Updownarrow$

HL for the degenerate action. $\Rightarrow$ HL for $H^*(X,k)$

<u>Rmk</u> In other types the difference is that P_i/P_{i+1} is not a projective space and the HL can be less immediate. For example if G of type D_n



and HL holds for any $p \neq 2$