W Coxeter group $W = \langle s \in S \mid (st)^{m_{st}} = 1, s^2 = 1 \rangle$

k field $m_{st} \in \{2,3,\dots,\infty\}$

$W \curvearrowright \mathfrak{h}/_{k}$ "reflection representation", i.e. $\{\alpha_s\} \subseteq \mathfrak{h}^*$, $\{\alpha_s^\vee\} \subseteq \mathfrak{h}$

$s(v) = v - \alpha_s^\vee(v)\alpha_s$ defines a representation of W

TECHNICAL ASSUMPTION reflection faithful i.e.

$T = \bigcup w S w^{-1}$ reflections

$\mathfrak{h}^y$ has codim $1 \iff y \in T$. (& good notion of positive roots)

CLASSICAL EXAMPLE $W = S_n \curvearrowright \overset{\mathfrak{h}}{\overset{\shortparallel}{k^n}}$, char $k \neq 2$

$t = (ij)$ $\alpha_t = \varepsilon_i - \varepsilon_j = \alpha_{s_i} + \dots + \alpha_{s_{j-1}}$

REFLECTION FAITHFUL $\Rightarrow$ $t \in T \overset{1:1}{\longleftrightarrow} \alpha_t \in \mathfrak{h}^*, \alpha_t^\vee \in \mathfrak{h}$

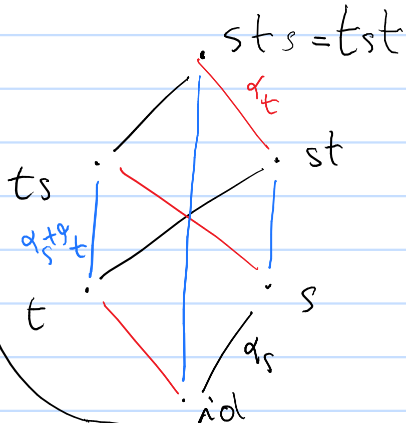
s.t. $t(v) = v -$

Out of $(W,\mathfrak{h})$ we construct the moment graph.

Vertices $x \in W$

Edges $x \xrightarrow{\alpha_t} tx$ $\forall x \in W$ $t \in T$ s.t. $tx \geq x$

E.g. $W = S_3 = \langle s,t \rangle$



$R = \mathrm{Sym}_k(\mathfrak{h}^*)$. $\deg(\mathfrak{h}^*) = 2$

Def A sheaf M on the moment graph is the data of

- M_x graded R-module $\forall x \in W$
- $M_{x \to tx}$ " " " $\forall$ edge s.t. $\alpha_t M_{x \to tx} = 0$
- morphism $M_x \xrightarrow{\rho_{x,tx}} M_{x \to tx} \xleftarrow{\rho_{tx,x}} M_{tx}$

EXAMPLE Constant sheaf $\mathcal{A}$: $\mathcal{A}_x = R$, $\mathcal{A}_{x-tx} = R/(\alpha_t)$

Global sections $\Gamma(\mathcal{M}) = \left\{ (m_x)_{x\in W},\ m_x \in \mathcal{M}_x \;\middle|\; \begin{array}{c} \mathcal{M}_x \qquad \mathcal{M}_{tx} \\ \rho_1 \searrow \quad \swarrow \rho_2 \\ \mathcal{M}_{x-tx} \end{array} \quad \rho_{x,tx}(m_x) = \rho_{tx,x}(m_{tx}) \right\}$

$\hat{Z} := \Gamma(\mathcal{A})$ is a ring

$\cup$

Z bounded sections (i.e. $\exists k$ s.t. $\deg(a_x) \leq k \ \forall x$)

Thm If W is a Weyl group (of Kac-Moody group G) then

$$Z \simeq H^*(G/B, \Bbbk).$$

For general Coxeter groups Z is free with basis indexed $\{\xi_x\}_{x\in W}$ and $\deg(\xi_x) = 2\ell(x)$

$$Z \cong \text{Kostant-Kumar dual nil Hecke ring.}$$

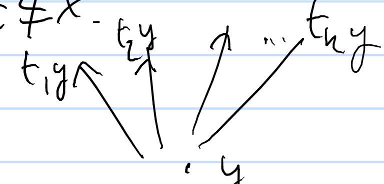
Moment graphs provide a combinatorial setting for studying intersection cohomology sheaves (Braden-MacPherson '01) or Parity sheaves (Fiebig-Williamson, '11)

Braden-MacPherson algorithm.

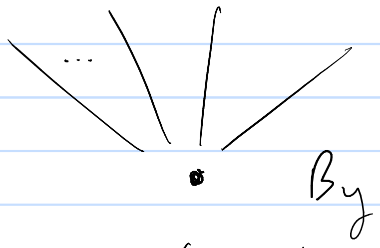
Construction of the canonical sheaves $\mathcal{B}(x)$

Start with $(\mathcal{B}(x))_x = R$, $(\mathcal{B}(x))_z = 0$ if $z \not\leq x$.

Assume $\mathcal{B}(x)_z$ defined for all $z > y$



$E = \{ y \to ty \}$ $\quad \mathcal{B}(x)_{y\to ty} = \mathcal{B}(x)_{ty}/(\alpha_t)$ for all arrows



E

$\Gamma(B_x, >y) \xrightarrow{\varphi} \bigoplus_{e\in E} B_e$

$(B_x)_y :=$ proj. cover of $\operatorname{Im}\varphi$

(in other words: $(B_x)_y$ is free and induces isomorph.

$$(B_x)_y \otimes_R \mathbb{k} \to (\operatorname{Im}\varphi) \otimes_R \mathbb{k})$$

Thm (Elias-Williamson) $\quad \operatorname{gr\,rk} (B_x)_y = P_{x,y}(q)$

KL polynomial

determine characters of irr. modules of reductive Lie algebras in the category $\mathcal{O}$

Thm (Fiebig-Williamson, '11)
W Weyl group, char $\mathbb{k} = p$

$\operatorname{gr\,rk} B(x)_y =$ pKL polynomials

determines character of tilting modules of reductive alg. group in char p

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BMP sheaves on the moment graphs

Soergel bimodules

HECKE CATEGORY

Parity sheaves (or IC sheaves in char 0)

Elias-Williamson diagrammatic category.

ONLY IF THERE IS TIME

<u>Soergel bimodules</u>: karoubian monoidal additive category of graded R-bimodule generated by $R \otimes_{R^s} R$ and shifts

Indecomposable Soergel bimodule are B_x, $x \in W$

$\Gamma(\mathcal{B}_x)$ is the indecomposable Soergel bimodule B_x

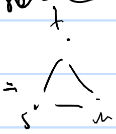
Notice that this gives a natural structure of Z-module on B_x which SBim do not see. $R \otimes R \to Z$ is not surjective
in general for infinite Coxeter group.

This becomes meaningful when we look at Soergel modules

B_x indecomp. Soergel bimodule

$\overline{B_x} := B_x \otimes_R \mathbb{R}$

<u>Thm (P.)</u> $\overline{B_x}$ is not indecomposable as a R-module but it is indecomposable as a Z-module

<u>Ex:</u> $w = stu\, tst$ in $\hat{A}_2$ = 

<u>Thm (P.)</u> $\hat{Z}$ is the center of the Hecke category.

COMBINATORICS & THE COEFFICIENT OF q (Assume $\mathfrak{h} = \mathbb{R}$)

$x \le y \in W$ $\quad [x,y] = \{ z \in W \mid x \le z \le w \}$ Bruhat interval

BMP algorithm $\Rightarrow$ Restriction of the moment graph to $[x,y]$
$\rightsquigarrow$ determines KL polynomial $P_{x,y}(q)$

<u>Conj</u> (Combinatorial invariance, Lusztig, Dyer '80s)
The KL polynomial $P_{x,y}(q)$ depends only on the graph type of $[x,y]$ (i.e. no label needed)

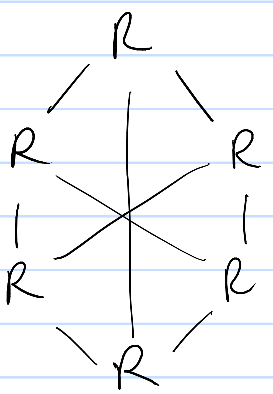
<u>Still open!</u> Some partial results

<u>Brenti-Caselli-Marietti '05</u> $\quad P_{e,x}(q)$ is a combinatorial invariant.

Conj holds if $\ell(x) - \ell(y) \le 4$
(≤ 8 in type A)

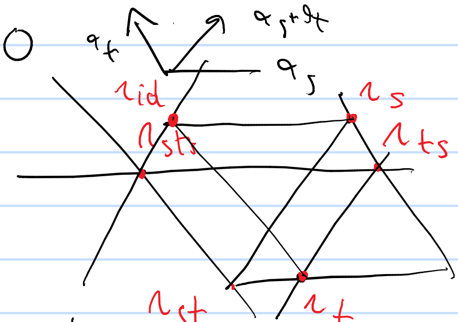
Still unknown even for the coefficient of q

$$q_{x,y} := \text{coeff. of } q \text{ in } P_{x,y}(q) = 1 + q_{x,y}\, q + \text{"higher terms"}$$



A section of degree 2 is the same thing as choosing six vectors $\eta_x \in \mathfrak{h}^*$
s.t. $\alpha_t \mid \eta_x - \eta_{tx}$

Say $r_{sts} = 0$



Want to make 2 points here.

1) r_{st}, r_{ts} determine the section. 2) The 3 lines always intersect in a point because of Pappus thm.

$$q_{x,y} = \operatorname{codim}\left(R_2 \xrightarrow{\pi} (B^{\delta_x})_2\right)$$

degree 2

$$= \dim \Gamma(B(x), >y)_2 - \dim \left\{ \begin{matrix} \pi\text{-extendable} \\ \text{sections} \end{matrix} \right\} =: V_{x,y}$$

A way to see $V_{x,y}$: if we fix the length of the top edges, when this gives rise to a solution?

In the example above $P_{sts,e}(q) = 1$ since all sections extend

$$c_{x,y} = \#\text{ coatoms in } [x,y]$$

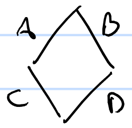
$$= \#\{x \geq z \geq y \mid \ell(z) = \ell(x) - 1\}$$

$$= \dim \Gamma(B_x, >x)^2$$

$$d_{x,y} = \dim V_{x,y}$$

Thm (Dyer '97) $q_{x,y} = c_{x,y} - d_{x,y}$ (BEFORE MOMENT GRAPHS WERE INTRODUCED!)

$$\begin{array}{c} E_{x,y} \\ \cup\!| \\ F \end{array}$$ set of edges of $[x,y]$

Let $\tilde{F} \supseteq F$ minimal such that whenever we have a square 

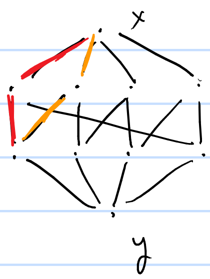
$\subseteq [x,y]$

$A, B \in \tilde{F} \Rightarrow C, D \in \tilde{F}$

We say F is square-generating if $\tilde{F} = E_{x,y}$

$g_{x,y} :=$ minimal size of a generating set.

EXAMPLES



Obs $g_{x,y} \geq d_{x,y}$ because fixing the length of the edges in F determines a section.

Obs. $g_{x,y} \leq c_{x,y}$. top edges always generates

$g_{x,y} \leq \ell(y) - \ell(x)$: every maximal chain generates (follows from shellability Bruhat interval)

Thm (P.) In type A $g_{x,y} = d_{x,y}$. (part of a longer joint project w/ Williamson)

Cor $q_{x,y} = c_{x,y} - g_{x,y}$ is combinatorial invariant in type A

Rmk Proof uses crucially generalized lifting property (Tsukerman–Williams '15)

$x \leq y$ $\exists$ reflection t s.t. $x < tx < y$ and $x < ty < y$ and such that $R_{x,y}(q) = (q-1) R_{x,ty} + q R_{tx,ty}$.