The quest for bases of the intersection cohomology of Schubert varieties

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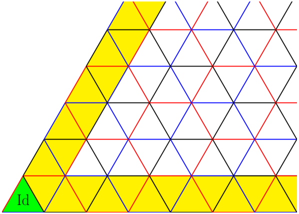
New Connections in Representation Theory

Mooloolaba

13th February 2019

Basis in type $\widetilde{A}_2$

W Weyl group of type $\widetilde{A}_2$.

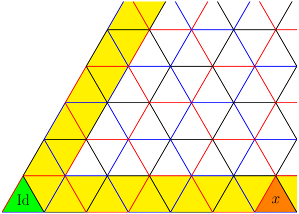


Over $\mathbb{Q}$ we have:

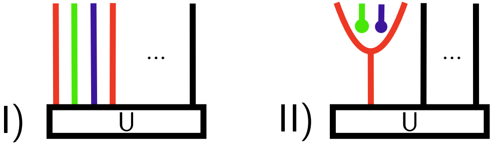
indecomposable objects in the Hecke category (aka indec. **Soergel bimodules**)	=	(equivariant) intersection cohomology of Schubert varieties in the affine flag variety $SL_3(\mathbb{C}((t)))/I$

$$B_x \cong IH^\bullet_T(X_x, \mathbb{Q})$$

Basis in type $\widetilde{A}_2$



A basis of $IH(X_x, \mathbb{Q})$ (x on the wall, $\ell(x)$ odd) is given by the following type of diagrams.



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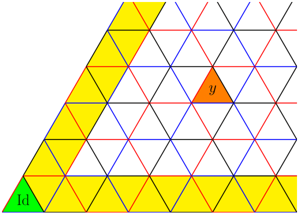
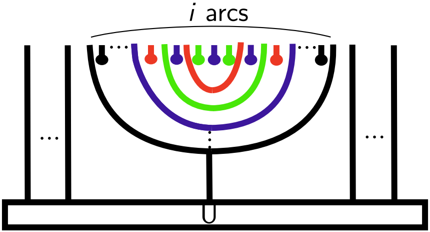


A basis of $IH(X_y, \mathbb{Q})$ (y out of the walls, y spherical) is given by the two following type of diagrams.



Our setting: Complex Grassmannian

$$\mathrm{Gr}(k, N) = \{k\text{-dimensional subspaces of } \mathbb{C}^N\}.$$

It is a smooth complex projective variety of dim $k(N-k)$.
What is $H^\bullet(\mathrm{Gr}(k, N), \mathbb{Q})$?

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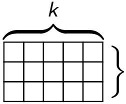
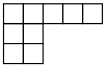
To any tableau $\lambda \subseteq$  $(N-k)$ it corresponds a **Schubert cell**.

Example



$$\mapsto C_\lambda := \mathrm{Im} \begin{pmatrix} * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \mathbb{C}^{\ell(\lambda)}$$

$\ell(\lambda) = \#\{\text{boxes in } \lambda\}$

Cohomology of Schubert varieties

Schubert cells give a cell decomposition:

$$\mathrm{Gr}(k, N) = \coprod_{\lambda} C_{\lambda}$$

Let $X_\mu = \overline{C_\mu}$ be a Schubert variety $\rightsquigarrow [X_\mu] \in H_{2\ell(\mu)}(\mathrm{Gr}(k, N))$.

$$H_\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q}[X_\lambda] \xrightarrow{\text{dualizing}} H^\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q}\mathcal{S}_\lambda$$

Taking dual basis of $\{[X_\lambda]\}$ we obtain a basis $\{\mathcal{S}_\lambda\}$,
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Via the pullback $i_\mu^* : H^\bullet(\mathrm{Gr}(k,N),\mathbb{Q}) \to H^\bullet(X_\mu,\mathbb{Q})$ we get

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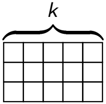
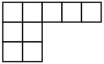
The T-equivariant story is analogous

Many equivalent combinatorial descriptions

The following sets are in bijections:

- Subtableaux of 

$\Big\}(N-k)$.



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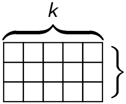
- Subtableaux of 

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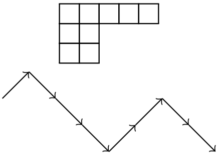
- Piece-wise linear paths from $(0, k)$ to $(N, N-k)$ with steps $(1, \pm 1)$.



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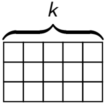
- Subtableaux of 

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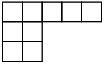

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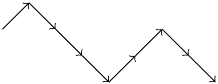

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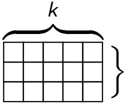
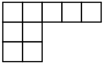
- Subtableaux of  $(N-k)$.



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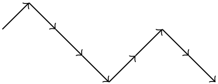

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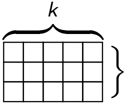
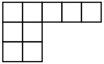
- Subtableaux of  $(N-k)$.

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  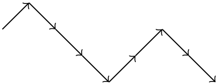

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 6 & 2 & 3 & 4 & 7 & 8 \end{pmatrix}$$

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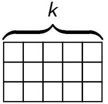
- Subtableaux of 

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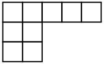

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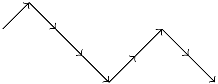

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Pieri's formula

The Schubert basis is an important tool to study $H^\bullet(\mathrm{Gr}(k,N),\mathbb{Q})$ and $H^\bullet(X_\mu,\mathbb{Q})$

Let $\mathcal{S}_\square \in H^2(\mathrm{Gr}(k,N),\mathbb{Q})$.

Pieri's formula

$$\mathcal{S}_\square \cdot \mathcal{S}_\lambda = \sum_{\substack{\mu \text{ tableau which can be} \\ \text{obtained by adding a box to } \lambda}} S_\mu$$

Example

$$\mathcal{S}_\square \cdot \mathcal{S}_{\ydiagram{2,2}} = \mathcal{S}_{\ydiagram{3,2}} + \mathcal{S}_{\ydiagram{2,2,1}}$$

Littlewood-Richardson Rule

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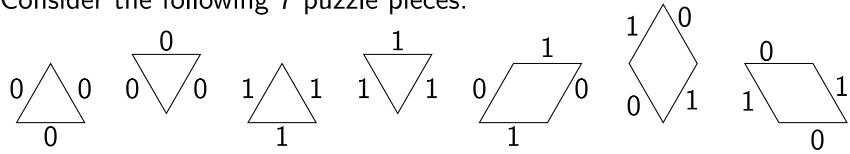

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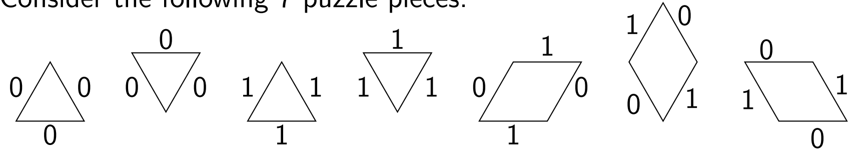
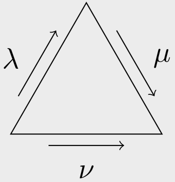


Theorem (Knutson-Tao-Woodward '04)

$$\mathcal{S}_\lambda \cdot \mathcal{S}_\mu = \sum_\nu c^\nu_{\lambda,\mu} \mathcal{S}_\nu$$

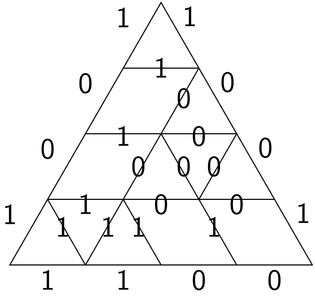
where $c^\nu_{\lambda,\mu}$ is the number of puzzles with boundary



Here we regard λ, μ, ν as elements of $\{0,1\}^N$.

Example of the puzzle rule

We have $\mathcal{S}_{1001} \cdot \mathcal{S}_{1001} = c_{1001,1001}^{1100} \mathcal{S}_{1100} \in H^8(\mathrm{Gr}(2,4))$
and $c_{1001,1001}^{1100} = 1$ because the only possible puzzle is:



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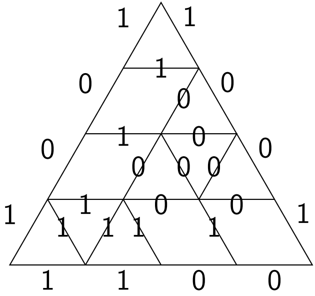
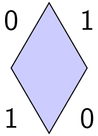


There is more! Adding a new "equivariant piece" one can compute $\mathcal{S}_\lambda \cdot \mathcal{S}_\mu \in H^\bullet_T(\mathrm{Gr}(k,N))$. (Knutson-Tao '05)



Intersection Cohomology of Schubert varieties

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Question

Is there some sort of "Schubert basis" for $IH^\bullet(X_\mu, \mathbb{Q})$?
Can we extend in a natural way $\{\mathcal{S}_\lambda\}$ to a basis of $IH^\bullet(X_\mu, \mathbb{Q})$?

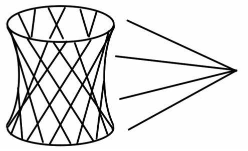
An example in $\mathrm{Gr}(2,4)$

Let $\mu = \ydiagram{2,1} \subseteq \ydiagram{2,2}$.

$X_\mu = \{V \subseteq \mathbb{C}^4 \mid \dim V = 2 \text{ and } \dim(V \cap \mathbb{C}^2) \geq 1\} \subseteq \mathrm{Gr}(2,4)$

X_μ is a singular variety of dim 3:

It is the projective cone over a non-degenerate quadric $Y \subseteq \mathbb{P}^3(\mathbb{C})$.



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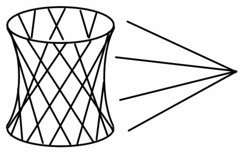

$H^\bullet(X_\mu) =$

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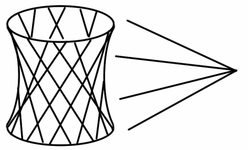

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	0		$\emptyset$			-3		$\emptyset$	

Kazhdan-Lusztig polynomial

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A quote by **Bernstein**: "...I would say that if you can compute a polynomial P for intersection cohomologies in some case without a computer, then probably there is a **small resolution** which gives it..."

Small resolutions

A resolution of singularities $p : \widetilde{X} \to X$ is said **small** if

$$\forall r > 0: \quad \mathrm{codim}\{x \in X \mid \dim p^{-1}(x) = r\} > 2r.$$

Schubert varieties in Grassmannians are "very special" Schubert varieties:

Theorem (Zelevinsky '83)

All the Schubert varieties in Grassmannians admit a small resolution of singularities.

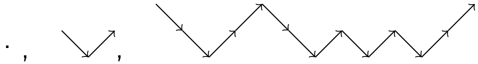
$$p \text{ small } \implies p_*\mathbb{Q}_{\widetilde{X}}[\dim X] \cong IC^\bullet(X, \mathbb{Q})$$
$$\implies H^\bullet(\widetilde{X}, \mathbb{Q}) = IH^{\bullet - \dim X}(X, \mathbb{Q}).$$

Dyck paths and strips

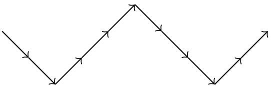
Definition

A **Dyck path** is a piecewise linear path consisting of the same number of segments $\searrow$ and $\nearrow$, such that it remains below the horizontal line.

Example



This is **not** a Dyck path:



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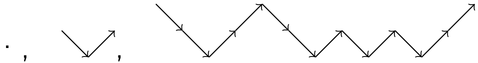
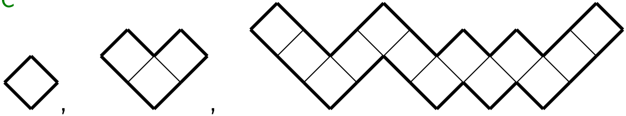
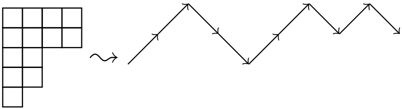


A **Dyck strip** is obtained by taking the unitary boxes, tilted by 45°, with center on the integral coordinates of a Dyck path

Example



We reinterpret a tableau as a path:



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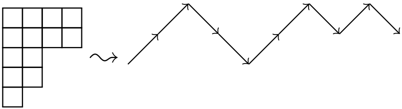
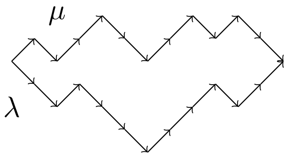
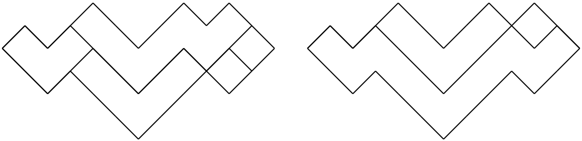


Let λ, μ be paths with $\lambda \leq \mu$. A **Dyck partition** is a partition of the region between λ and μ into Dyck strips.

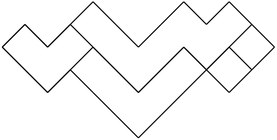
Example



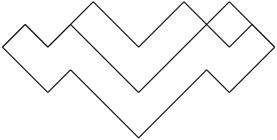


Dyck partitions of Type 1

A Dyck partition **P** is said of **type 1** if:
whenever a strip D contains a box just below a box in a strip C, then every box just above a box in D is in C.



Type 1



Not Type 1

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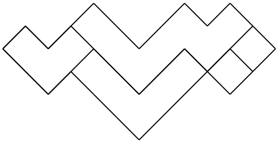

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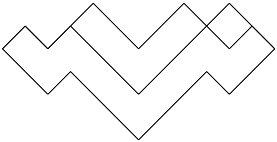

Not Type 1

We can reformulate LS results using **Dyck partitions**
$|\mathbf{P}|$ = number of strips in **P**.

Theorem (Shigechi - Zinn-Justin '12)

Dyck partitions of Type 1 describes KL polynomials.

$$\sum_{\substack{\mathbf{P} \text{ of type 1} \\ \text{between } \lambda \text{ and } \mu}} v^{|\mathbf{P}|} = h_{\lambda,\mu}(v)$$

Singular Soergel bimodules

$R = \mathbb{Q}[x_1, x_2, \ldots, x_N] \curvearrowleft S_N$ acts by permutations.
R^{S_N} is the subring of invariants. Then:

$$H^{\bullet}_{T}(\mathrm{Gr}(k, N)) = R \otimes_{R^{S_N}} R^{S_k \times S_{N-k}}.$$

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We regard $IH_\lambda := IH_T^\bullet(X_\lambda, \mathbb{Q})$ via the inclusion $X_\lambda \hookrightarrow \mathrm{Gr}(k,N)$ as a $H_T^\bullet(\mathrm{Gr}(k,N))$-module (or as a $(R, R^{S_k \times S_{N-k}})$-bimodule).
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Let $\lambda < \mu$. If we quotient out all morphisms factoring through

$$IH_\lambda \to IH_\nu \to IH_\mu \text{ for some } \nu < \lambda$$

we have

$$\mathrm{Hom}^\bullet_{\not<\lambda}(IH_\lambda, IH_\mu) \cong \bigoplus_i R(-i)^{m_i} \text{ with } \sum_i m_i v^i = h_{\lambda,\mu}(v)$$

Morphisms of degree one

$$\mathrm{Hom}^1(IH_\lambda, IH_\mu) \cong \begin{cases} \mathbb{Q} & \text{if } \lambda \text{ and } \mu \text{ differ by a Dyck strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\{\text{Dyck strips}\} \to \{\text{morphisms of degree 1}\}$$

$$D \mapsto f_D$$

The map f_D can be explicitly constructed.

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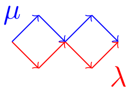
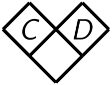
Naive idea:

$$\left\{ \begin{matrix} \text{Dyck partition} \\ \text{with } m \text{ elements} \end{matrix} \right\} \to \{\text{morphisms of degree } m\}$$

$$\mathbf{P} = \{D_1, D_2, \ldots, D_m\} \mapsto f_{\mathbf{P}} := f_{D_1} \circ f_{D_2} \circ \ldots f_{D_m}$$

Dyck strips do not commute



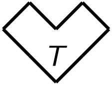


Then

$$f_C \circ f_D \neq f_D \circ f_C \in \mathrm{Hom}^2(IH_\lambda, IH_\mu).$$

and $f_C \circ f_D - f_D \circ f_C$ is a non-zero map factoring through

$$IH_\lambda \to IH_\emptyset \xrightarrow{f_T} IH_\mu$$ where



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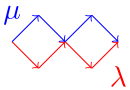

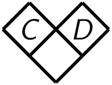

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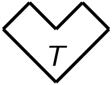


In general, $f_{\mathbf{P}} := f_{D_1} \circ f_{D_2} \circ \ldots f_{D_k}$ **not** well defined.

A partial order on Dyck partitions

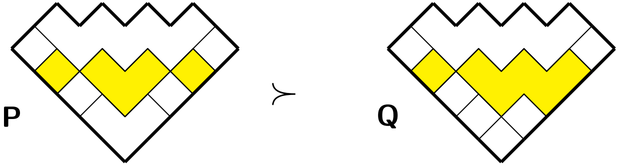
Let **P** and **Q** be Dyck partition, $\mathbf{P} \neq \mathbf{Q}$.

$\mathbf{P}(h) :=$ set of strips of height h in **P**.

$h_0 :=$ largest index such that $\mathbf{P}(h_0) \neq \mathbf{Q}(h_0)$.

Then $\mathbf{P} \succ \mathbf{Q}$ if $\mathbf{P}(h_0)$ is finer than $\mathbf{Q}(h_0)$,
i.e. if every strip of $\mathbf{P}(h_0)$ is contained in a strip of $\mathbf{Q}(h_0)$.

Example



Construction of bases on morphisms spaces

Theorem (P. '19)

If $\mathbf{P} = \{D_1, D_2, \dots, D_m\}$ is a Dyck partition between λ and μ, then the map

$$f_{\mathbf{P}} = f_{D_1} \circ f_{D_2} \circ \dots \circ f_{D_m} \in \mathrm{Hom}^m_{\not<\mu}(IH_\mu, IH_\lambda).$$

does not depend on the order up to smaller terms wrt $\prec$, i.e. up to something contained in $span\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P}\rangle$.

The set

$$\left\{ f_{\mathbf{P}} \mid \begin{array}{c} \mathbf{P} \text{ Dyck partition of type 1} \\ \text{between } \lambda \text{ and } \mu \end{array} \right\}$$

is a basis of $\mathrm{Hom}_{\not<\lambda}(IH_\lambda, IH_\mu)$ over R, for any choice of the order of the strips in $\mathbf{P}$.

Construction of the bases of Intersection Cohomology

$\mathcal{S}_{id}$ is the unity of the cohomology ring $H_T(X_\lambda)$.
Let $F_{\mathbf{P}} := f_{\mathbf{P}}(\mathcal{S}_{id})$.

Corollary

The element $F_{\mathbf{P}} \in IH_\mu^{-\ell(\mu)+2|\mathbf{P}|}$ does not depend on the order chosen, up to smaller elements wrt $\prec$.

The set

$$\left\{ F_{\mathbf{P}} \mid \begin{array}{c} \mathbf{P} \text{ Dyck partition of type 1} \\ \text{between } \lambda \text{ and } \mu \text{, for some } \lambda \leq \mu \end{array} \right\}$$

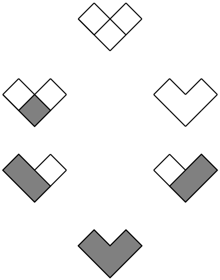
is a basis of IH_μ over R, for any choice of the order of the strips in $\mathbf{P}$.

Back to the example in Gr$(2,4)$

$$\mu = \yng(2,1) \subseteq \yng(2,2).$$

3		$\yng(2,1)$	
1	$\yng(2)$		$\yng(1,1)$
-1	$\yng(1)$		?
-3		$\emptyset$	

$$= IH^{\bullet}(X_{\mu}, \mathbb{Q}) =$$



Comparison with the Schubert basis

$\langle -, - \rangle_\mu$ Poincaré pairing on IH_μ.

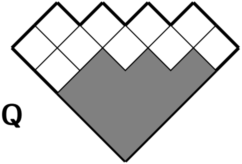
P Dyck partition of type 1 between ν and μ. For any $\lambda \leq \mu$:

$$\langle \mathcal{S}_\lambda, F_{\mathbf{P}} \rangle = \begin{cases} 1 & \text{if } \nu = \lambda \text{ and } \mathbf{P} \text{ only consists of single boxes} \\ 0 & \text{otherwise} \end{cases}$$

Hence: if $\{F^*_{\mathbf{P}}\}$ is the dual basis to $\{F_{\mathbf{P}}\}$ then

$$\mathcal{S}_\lambda = F^*_{\mathbf{Q}} \text{ where } \mathbf{Q} \text{ consists only of single boxes.}$$

Q



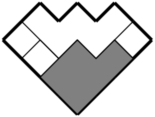
Pieri's formula in intersection cohomology

Proposition

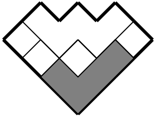
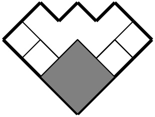
$$\mathcal{S}_{\square} \cdot F_{\mathbf{P}} = \sum_{\substack{C \text{ box that can} \\ \text{be added to } \mathbf{P}}} F_{\mathbf{P} \cup \{C\}}$$

where the order on $\mathbf{P} \cup \{C\}$ is the same order on $\mathbf{P}$ plus C at the beginning.

Example

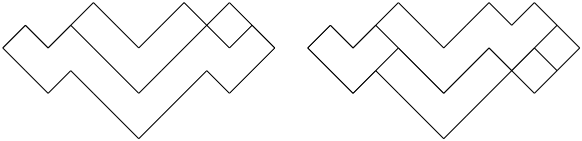
If $\mathbf{P} =$  then

$$\mathcal{S}_{\square} \cdot F_{\mathbf{P}} = F_{\mathbf{Q}_1} + F_{\mathbf{Q}_2}$$

where $\mathbf{Q}_1 =$  and $\mathbf{Q}_2 =$ 

Dyck partitions of Type 2

A Dyck partition **P** is said of **type 2** if:
for any strip D that contains a box just below, SW or SE o a box in a strip C, then every box just below, SW or SE a box in C belongs either to D or C.



Type 2 Not Type 2

Remark

Between any two paths λ, μ there exists **at most one** Dyck partition of type 2.

Inverse KL polynomials

Inverse KL polynomials are related to the ordinary KL polynomials by the **inversion formula**:

$$\sum_{\mu} (-1)^{\ell(\mu)-\ell(\nu)} h_{\lambda,\mu}(v) g_{\mu,\nu}(v) = \delta_{\lambda,\nu}$$

Theorem (Brenti '02)

Dyck partitions of Type 2 describe **inverse** KL polynomials

$$\sum_{\substack{\mathbf{P} \text{ of type 2} \\ \text{between } \lambda \text{ and } \mu}} v^{|\mathbf{P}|} = g_{\lambda,\mu}(v)$$

Singular Rouquier Complexes

We can construct a complex of singular Soergel bimodules E_μ :

$$0 \to \bigoplus_{\lambda \in \Lambda_i} IH_\lambda(-i) \to \ldots \to \bigoplus_{\lambda \in \Lambda_1} IH_\lambda(-1) \to IH_\mu \to 0$$

which is **exact** everywhere but in the term IH_μ.

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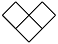
which is **exact** everywhere but in the term IH_μ.

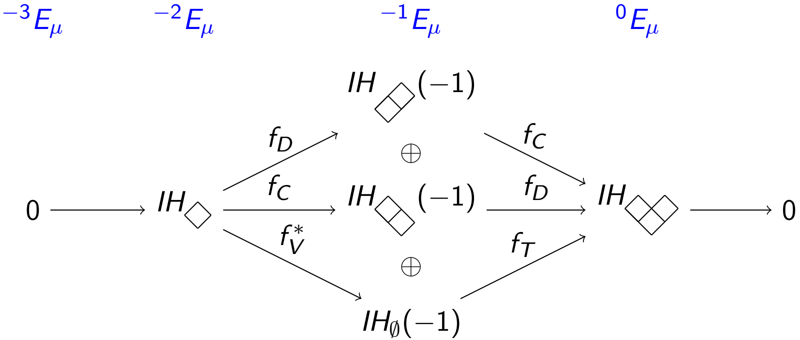
Here $\Lambda_i = \left\{ \mu \leq \lambda \mid \begin{array}{c} \text{exists Dyck partition } \mathbf{P} \\ \text{of type 2 between } \lambda \text{ and } \mu \text{ with } |\mathbf{P}| = i \end{array} \right\}.$

$$IH_\lambda(-i) \text{ occurs in } E_\mu \quad \iff \quad g_{\lambda,\mu}(v) = v^i$$

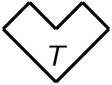
E_μ is called **singular Rouquier complex**.

Example of a singular Rouquier complex

Let $\mu =$ . The singular Rouquier complex E_μ is:



where 

and 

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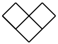
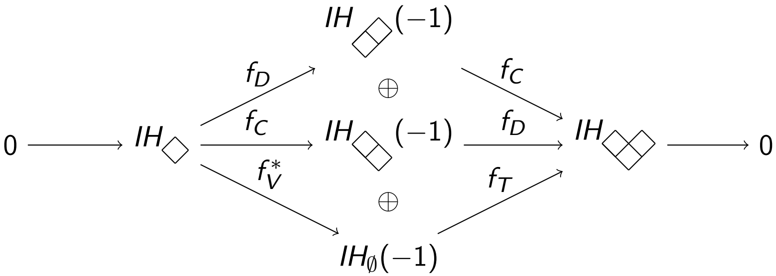
Let $\mu =$ . The singular Rouquier complex E_μ is:

${}^{-3}E_\mu$ ${}^{-2}E_\mu$ ${}^{-1}E_\mu$ ${}^{0}E_\mu$



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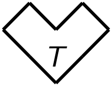
and 

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From $d^2 = 0$ follows $f_D \circ f_C - f_C \circ f_D = f_T \circ f_V^*$ (up to scalar).

Thanks for your attention!