Extending Schubert calculus to intersection cohomology

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RepNet Virtual Seminar

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Part 1:
Schubert varieties in complex Grassmannians

Complex Grassmannian

$$\mathrm{Gr}(k, N) = \{k\text{-dimensional subspaces of } \mathbb{C}^N\}.$$

It is a smooth complex projective variety of dim $k(N-k)$.
What is $H^\bullet(\mathrm{Gr}(k, N), \mathbb{Q})$?

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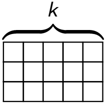
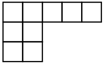
To any tableau $\lambda \subseteq$ 

$\}(N-k)$ it corresponds a **Schubert cell**.

Example

 $\mapsto C_\lambda := \mathrm{Im}\begin{pmatrix} * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \mathbb{C}^{\ell(\lambda)}$

$\ell(\lambda) = \#\{\text{boxes in } \lambda\}$

Cohomology of Schubert varieties

Schubert cells give a cell decomposition:

$$\mathrm{Gr}(k, N) = \coprod_{\lambda} C_\lambda$$

Let $X_\mu = \overline{C_\mu}$ be a Schubert variety $\rightsquigarrow [X_\mu] \in H_{2\ell(\mu)}(\mathrm{Gr}(k, N))$.

$$H_\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q}[X_\lambda] \xrightarrow{\text{dualizing}} H^\bullet(\mathrm{Gr}(k, N), \mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q}\mathcal{S}_\lambda$$

Taking dual basis of $\{[X_\lambda]\}$ we obtain a basis $\{\mathcal{S}_\lambda\}$,
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Let $i_\mu : X_\mu \hookrightarrow \mathrm{Gr}(k,N)$ be the inclusion.
Via the pullback $i_\mu^* : H^\bullet(\mathrm{Gr}(k,N),\mathbb{Q}) \to H^\bullet(X_\mu,\mathbb{Q})$ we get

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The T-equivariant story is analogous

Pieri's formula

The Schubert basis is an important tool to study $H^\bullet(\mathrm{Gr}(k,N),\mathbb{Q})$ and $H^\bullet(X_\mu,\mathbb{Q})$

Let $\mathcal{S}_\square \in H^2(\mathrm{Gr}(k,N),\mathbb{Q})$.

Pieri's formula

$$\mathcal{S}_\square \cdot \mathcal{S}_\lambda = \sum_{\substack{\mu \text{ tableau which can be} \\ \text{obtained by adding a box to } \lambda}} S_\mu$$

Example

$$\mathcal{S}_\square \cdot \mathcal{S}_{\ydiagram{2,2}} = \mathcal{S}_{\ydiagram{3,2}} + \mathcal{S}_{\ydiagram{2,2,1}}$$

Littlewood-Richardson Rule

We can write

$$\mathcal{S}_\lambda \cdot \mathcal{S}_\mu = \sum_\nu c^\nu_{\lambda\mu} \mathcal{S}_\nu.$$

where $c^\nu_{\lambda\mu} \in \mathbb{N}$ are the **Littlewood-Richardson coefficients.**

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There are many combinatorial models for $c^\nu_{\lambda\mu}$.

- Littlewood-Richardson tableaux (i.e. semistandard skew-tableau of shape λ/μ with content ν whose word is a lattice.)
- Berenstein–Zelevinsky patterns
- Knutson-Tao honeycombs
- Knutson-Tao-Woodward puzzles
- ...

Intersection Cohomology of Schubert varieties

If X_μ is singular, Poincaré duality fails.

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We can embed the cohomology into the **intersection cohomology**:

$$H^\bullet(X_\mu, \mathbb{Q}) \subseteq IH^{\bullet-\ell(\mu)}(X_\mu, \mathbb{Q}).$$

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Question

Is there some sort of "Schubert basis" for $IH^\bullet(X_\mu, \mathbb{Q})$?
Can we extend in a natural way $\{\mathcal{S}_\lambda\}$ to a basis of $IH^\bullet(X_\mu, \mathbb{Q})$?

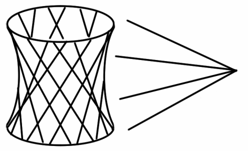
An example in $\mathrm{Gr}(2,4)$

Let $\mu = \ydiagram{2,1} \subseteq \ydiagram{2,2}$.

$X_\mu = \{V \subseteq \mathbb{C}^4 \mid \dim V = 2 \text{ and } \dim(V \cap \mathbb{C}^2) \geq 1\} \subseteq \mathrm{Gr}(2,4)$

X_μ is a singular variety of dim 3:

It is the projective cone over a non-degenerate quadric $Y \subseteq \mathbb{P}^3(\mathbb{C})$.



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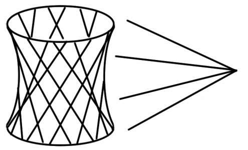

$$H^\bullet(X_\mu) = \begin{array}{cccc} 6 & & \ydiagram{2,1} & \\ 4 & \ydiagram{2} & & \ydiagram{1,1} \\ 2 & & \ydiagram{1} & \\ 0 & & \emptyset & \end{array}$$

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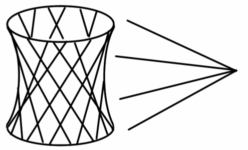

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$IH^\bullet(X_\mu) =$

3		$\ydiagram{2,1}$	
1	$\ydiagram{2}$		$\ydiagram{1,1}$
-1	$\ydiagram{1}$		?
-3		$\emptyset$	

Kazhdan-Lusztig polynomial

The (Grassmannian) **Kazhdan-Lusztig polynomial** can be defined as the Poincaré polynomials of the stalks of the IC sheaf.

$$h_{\lambda,\mu}(v) = \sum_i \dim IC_{C_\lambda}^{-i-\ell(\mu)}(X_\mu, \mathbb{Q}) v^i.$$

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A quote by **Bernstein**: "...I would say that if you can compute a polynomial P for intersection cohomologies in some case without a computer, then probably there is a **small resolution** which gives it..."

Small resolutions

A resolution of singularities $p : \widetilde{X} \to X$ is said **small** if

$$\forall r > 0: \quad \mathrm{codim}\{x \in X \mid \dim p^{-1}(x) = r\} > 2r.$$

Schubert varieties in Grassmannians are "very special" Schubert varieties:

Theorem (Zelevinsky '83)

All the Schubert varieties in Grassmannians admit a small resolution of singularities.

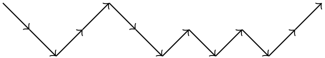
$$p \text{ small} \implies p_*\mathbb{Q}_{\widetilde{X}}[\dim X] \cong IC^\bullet(X, \mathbb{Q})$$
$$\implies H^\bullet(\widetilde{X}, \mathbb{Q}) = IH^{\bullet - \dim X}(X, \mathbb{Q}).$$

Dyck paths and strips

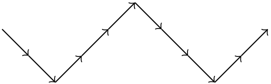
Definition

A **Dyck path** is a piecewise linear path consisting of the same number of segments ↘ and ↗, such that it remains below the horizontal line.

Example

· , , 

This is **not** a Dyck path:



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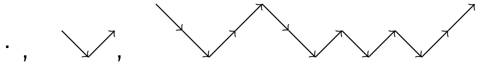
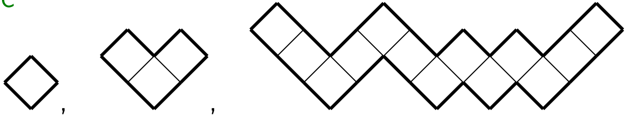
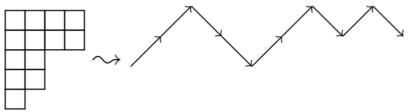


A **Dyck strip** is obtained by taking the unitary boxes, tilted by 45°, with center on the integral coordinates of a Dyck path

Example



We reinterpret a tableau as a path:



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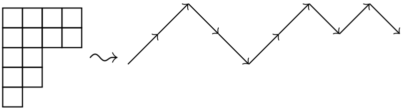
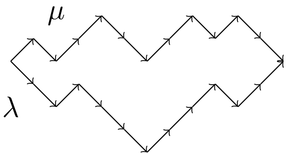
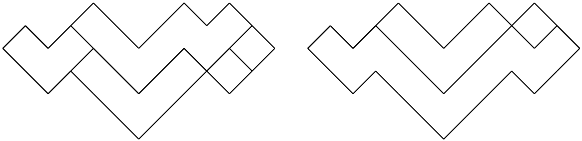


Let λ, μ be paths with $\lambda \leq \mu$. A **Dyck partition** is a partition of the region between λ and μ into Dyck strips.

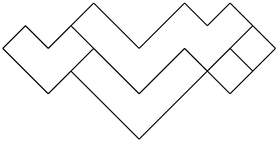
Example





Dyck partitions of Type 1

A Dyck partition P is said of **type 1** if
given a strip D, only bigger strips are placed on top of it.



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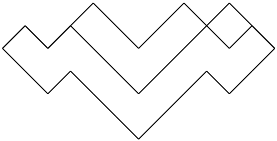

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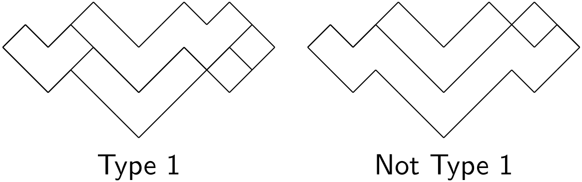

We can reformulate LS results using **Dyck partitions**
$|\mathrm{P}| =$ number of strips in P.

Theorem (Shigechi - Zinn-Justin '12)

Dyck partitions of Type 1 describes KL polynomials.

$$\sum_{\substack{\text{P of type 1} \\ \text{between } \lambda \text{ and } \mu}} v^{|\mathrm{P}|} = h_{\lambda,\mu}(v)$$

Singular Soergel bimodules

$R = \mathbb{Q}[x_1, x_2, \ldots, x_N] \curvearrowleft S_N$ acts by permutations.
R^{S_N} is the subring of invariants. Then:

$$H_T^\bullet(\mathrm{Gr}(k, N)) = R \otimes_{R^{S_N}} R^{S_k \times S_{N-k}}.$$

where $T = (\mathbb{C}^*)^N$ acts on $\mathrm{Gr}(k, N)$.

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We regard $IH_\lambda := IH^\bullet_T(X_\lambda, \mathbb{Q})$ via the inclusion $X_\lambda \hookrightarrow \mathrm{Gr}(k,N)$ as a $H^\bullet_T(\mathrm{Gr}(k,N))$-module (or as a $(R, R^{S_k \times S_{N-k}})$-bimodule).
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Existence of small resolutions $\implies$ Every IH_λ can be realized as

$$IH_\lambda \cong R \otimes_{R^{I_1}} R^{I_2} \otimes_{R^{I_3}} R^{I_4} \otimes \ldots \otimes R^{S_k \times S_{N-k}}$$

for some parabolic subgroups $I_j \subset S_N$.

Soergel bimodules and KL polynomials

Let $\lambda \subset \mu$. If we quotient out all morphisms factoring through

$$IH_\lambda \to IH_\nu \to IH_\mu \text{ for some } \nu \subset \lambda$$

we have

$$\mathrm{Hom}^\bullet_{\not<\lambda}(IH_\lambda, IH_\mu) \cong \bigoplus_i R(-i)^{m_i} \text{ with } \sum_i m_i v^i = h_{\lambda,\mu}(v)$$

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Example:

$$\mathrm{Hom}^1(IH_\lambda, IH_\mu) \cong \begin{cases} \mathbb{Q} & \text{if } \lambda \text{ and } \mu \text{ differ by a Dyck strip} \\ 0 & \text{otherwise} \end{cases}$$

Morphisms of degree one

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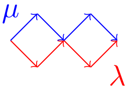
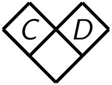
Naive idea:

$$\left\{ \begin{array}{c} \text{Dyck partition} \\ \text{with } m \text{ elements} \end{array} \right\} \to \{\text{morphisms of degree } m\}$$

$$\mathsf{P} = \{D_1, D_2, \ldots, D_m\} \mapsto f_{\mathsf{P}} := f_{D_1} \circ f_{D_2} \circ \ldots f_{D_m}$$

Dyck strips do not commute





Then

$$f_C \circ f_D \neq f_D \circ f_C \in \mathrm{Hom}^2(IH_\lambda, IH_\mu).$$

and $f_C \circ f_D - f_D \circ f_C$ is a non-zero map factoring through

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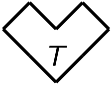

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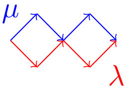

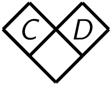

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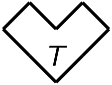


In general, $f_{\mathrm{P}} := f_{D_1} \circ f_{D_2} \circ \ldots f_{D_k}$ **not** well defined.

A partial order on Dyck partitions

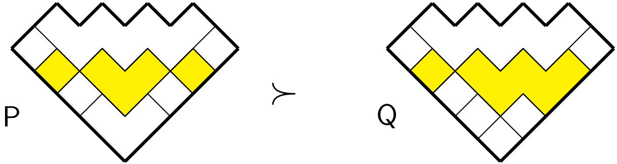
Let P and Q be Dyck partition, $P \neq Q$.

$P(h) :=$ set of strips of height h in P.

$h_0 :=$ largest index such that $P(h_0) \neq Q(h_0)$.

Then $P \succ Q$ if $P(h_0)$ is finer than $Q(h_0)$,
i.e. if every strip of $P(h_0)$ is contained in a strip of $Q(h_0)$.

Example



Construction of bases on morphisms spaces

Theorem (P. '19)

If $\mathsf{P} = \{D_1, D_2, \dots, D_m\}$ is a Dyck partition between λ and μ, then the map

$$f_{\mathsf{P}} = f_{D_1} \circ f_{D_2} \circ \dots \circ f_{D_m} \in \mathrm{Hom}^m_{\not<\mu}(IH_\mu, IH_\lambda).$$

does not depend on the order up to smaller terms wrt $\prec$,
i.e. up to something contained in $span\langle f_{\mathsf{Q}} \mid \mathsf{Q} \prec \mathsf{P}\rangle$.

The set

$$\left\{ f_{\mathsf{P}} \mid \begin{array}{c} \mathsf{P} \text{ Dyck partition of type 1} \\ \text{between } \lambda \text{ and } \mu \end{array} \right\}$$

is a basis of $\mathrm{Hom}_{\not<\lambda}(IH_\lambda, IH_\mu)$ over R, for any choice of the order of the strips in P.

$\mathcal{S}_{id}$ is the unity of the cohomology ring $H_T(X_\lambda)$.
Let $F_{\mathsf{P}} := f_{\mathsf{P}}(\mathcal{S}_{id})$.

Corollary

The element $F_{\mathsf{P}} \in IH_\mu^{-\ell(\mu)+2|\mathsf{P}|}$ does not depend on the order chosen, up to smaller elements wrt $\prec$.

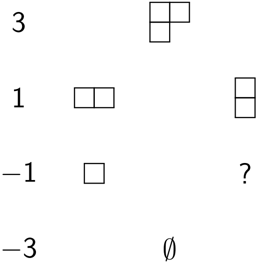
The set

$$\left\{ F_{\mathsf{P}} \mid \begin{array}{c} \mathsf{P} \text{ Dyck partition of type 1} \\ \text{between } \lambda \text{ and } \mu, \text{ for some } \lambda \leq \mu \end{array} \right\}$$

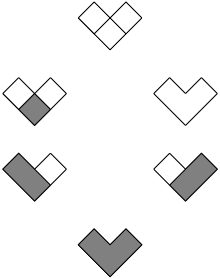
is a basis of IH_μ over R, for any choice of the order of the strips in P.

Back to the example in Gr(2, 4)

$\mu =$  $\subseteq$ .



$= IH^\bullet(X_\mu, \mathbb{Q}) =$



Comparison with the Schubert basis

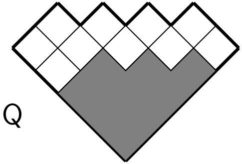
$\langle -, - \rangle_\mu$ Poincaré pairing on IH_μ.

P Dyck partition of type 1 between ν and μ. For any $\lambda \leq \mu$:

$$\langle \mathcal{S}_\lambda, F_{\mathsf{P}} \rangle = \begin{cases} 1 & \text{if } \nu = \lambda \text{ and P only consists of single boxes} \\ 0 & \text{otherwise} \end{cases}$$

Hence: if $\{F^*_{\mathsf{P}}\}$ is the dual basis to $\{F_{\mathsf{P}}\}$ then

$$\mathcal{S}_\lambda = F^*_{\mathsf{Q}} \text{ where Q consists only of single boxes.}$$

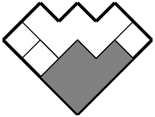


Pieri's formula in intersection cohomology

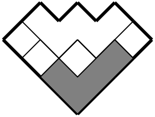
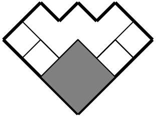
Proposition $$\mathcal{S}_{\square} \cdot F_{\mathrm{P}} = \sum_{\substack{C \text{ box that can} \\ \text{be added to } \mathrm{P}}} F_{\mathrm{P}\cup\{C\}}$$

where the order on $\mathrm{P} \cup \{C\}$ *is the same order on* P *plus* C *at the beginning.*

Example

If $\mathrm{P} =$  then

$$\mathcal{S}_{\square} \cdot F_{\mathrm{P}} = F_{\mathrm{Q}_1} + F_{\mathrm{Q}_2}$$

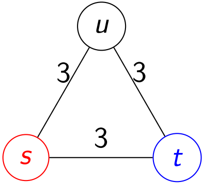
where $\mathrm{Q}_1 =$  and $\mathrm{Q}_2 =$ 

Part 2: the group $\widetilde{A}_2$

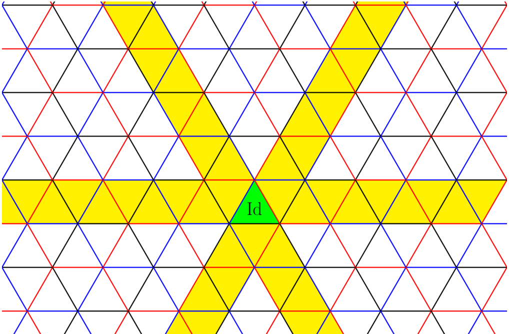
joint with Nicolas Libedinsky

W affine Weyl group of type $\widetilde{A}_2$.

Coxeter diagram:



$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = Id \rangle.$$



Schubert varieties for $\widetilde{A}_2$

Let $\mathcal{F}l = SL_3\left(\mathbb{C}(\!(t)\!)\right)/\mathrm{Iw}$ be the *affine flag variety* where

$$\mathrm{Iw} = \left\{ \begin{pmatrix} * & * & * \\ a & * & * \\ b & c & * \end{pmatrix} \in SL_3(\mathbb{C}[[t]]) \mid a, b, c \in t\mathbb{C}[[t]] \right\}$$

Bruhat decomposition:

$$\mathcal{F}l^\circ = \coprod_{y \in W} \mathrm{Iw} \cdot y \, \mathrm{Iw}$$

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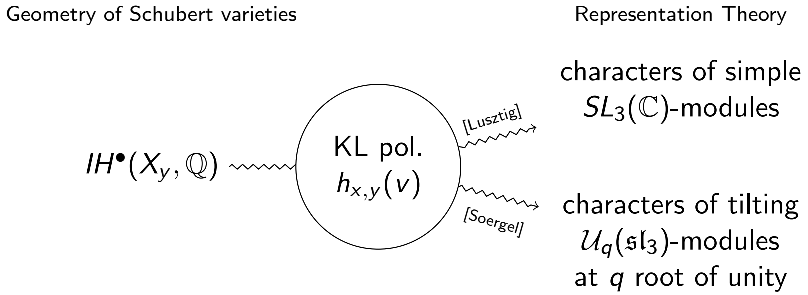
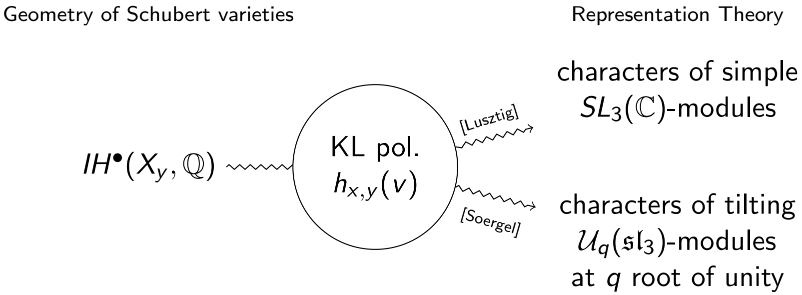
The Schubert variety $X_y = \overline{\mathrm{Iw} \cdot y \mathrm{Iw}}$ is a projective (usually singular) algebraic variety of dim $\ell(y)$.
As in the first part, we are interested in $IH^{\bullet}(X_y)$.

Connections with representation theory



Connections with representation theory



Also in this case we have

$$H^\bullet(X_y, \mathbb{Q}) \subseteq IH^{\bullet-\ell(y)}(X_y, \mathbb{Q})$$

We want to extend the Schubert basis from $H^\bullet(X_y, \mathbb{Q})$ to $IH^\bullet(X_y, \mathbb{Q})$.

Soergel bimodules for $\widetilde{A}_2$

Let $R = \mathbb{Q}[\alpha_s, \alpha_t, \alpha_u]$ (it is a polynomial ring with $\deg(\alpha_i) = 2$).

There is an action of W on R associated to the Cartan matrix

$$(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Example: $s(\alpha_u) = \alpha_u - (\alpha_s, \alpha_u)\alpha_s = \alpha_u + \alpha_s$

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For every reduced expression $y = st \dots u$ we can realize $IH_y := IH^\bullet_T(X_y, \mathbb{Q})$ as a direct summand of the Bott-Samelson bimodule

$$BS(\underline{st \dots u}) = R \otimes_{R^s} R \otimes_{R^t} R \otimes \dots \otimes_{R^u} R(\ell(y))$$

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Example: IH_{sts} is a summand (as a R-bimodule) of

$$BS(\underline{sts}) = R \otimes_{R^s} R \otimes_{R^t} R \otimes_{R^s} R(3) \cong IH_{sts} \oplus IH_s$$

Light leaves morphisms

Morphisms between Bott-Samelson bimodules are well understood.
Let $y \leq w$. Let $\underline{w} = s_1 s_2 \dots s_k$ reduced expression.
For $e \in \{0, 1\}^k$ we write $\underline{w}^e := s_1^{e_1} s_2^{e_2} \dots s_k^{e_k}$
Example: $\underline{sts}^{100} = s^1 t^0 s^0 = s$.

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Theorem (Libedinsky '07, Elias-Williamson '13)

$$\mathrm{Hom}_{\not< y}(BS(\underline{y}), BS(\underline{w})) = \bigoplus_{\underline{w}^e = y} \mathbb{Q} LL_{\underline{w},e}$$

$LL_{\underline{w},e}$ is a map that can be constructed algorithmically out of $\underline{w}$ and e.

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Question

Can we find a subset of $\{LL_{\underline{w},e}\}$ that gives a basis when restricted to the summand $IH_w \subset BS(\underline{w})$?

Morphisms diagrammatically

To depict morphisms between Bott-Samelson, it is very convenient to use diagrams.

$$\begin{array}{c} BS(\underline{s}) = R \otimes_{R^s} R \\ \uparrow \\ BS(\emptyset) = R \end{array}$$

$$\begin{array}{c} \tfrac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \\ \mapsfrom \\ 1 \end{array}$$

$$\begin{array}{c} BS(\underline{ss}) = R \otimes_{R^s} R \otimes_{R^s} R \\ \uparrow \\ BS(\underline{s}) = R \otimes_{R^s} R \end{array}$$

$$\begin{array}{c} f \otimes 1 \otimes g \\ \mapsfrom \\ f \otimes g \end{array}$$

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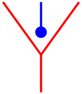
Example

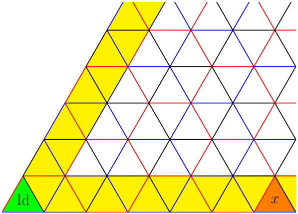


depicts a morphism $BS(\underline{s}) \to BS(\underline{sts})$

(which is actually $LL_{\underline{sts},100}$).

Relations on light leaves on the walls

The elements on the walls (in yellow in the picture) have a **unique reduced expression** of the form *stustu* ...



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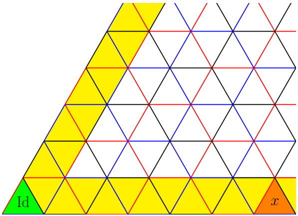

For x on the wall, post-composing with idempotent e_x of IH_x

$$BS(\underline{y}) \xrightarrow{LL_{\underline{x},e}} BS(\underline{x}) \xrightarrow{e_x} IH_x$$

we obtain the following relations on the set $\{LL_{\underline{x},e}\}$.

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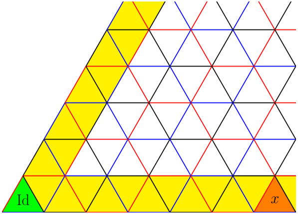

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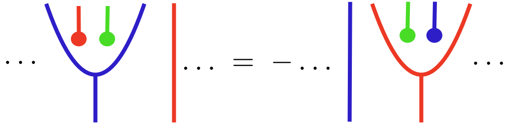
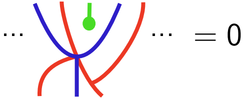
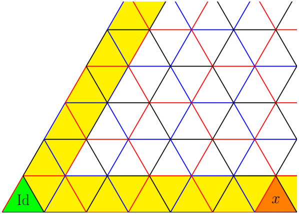
- 

- 

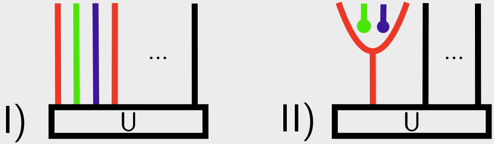
- Everything with two cups vanishes.

Basis in type $\widetilde{A}_2$ (on the walls)



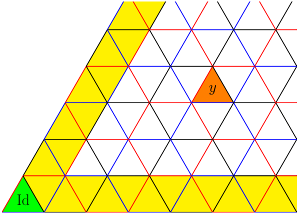
Theorem (Libedinsky-P. '20)

A basis of IH_x (x on the wall, $\ell(x)$ odd) is given by the following type of light leaves.



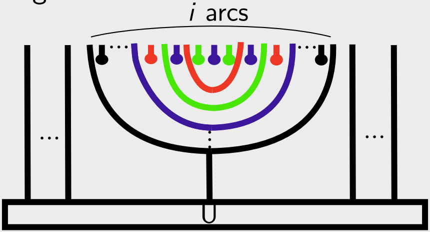
where the box with U is a light leaf containing only dots (no trivalent vertices allowed).

Basis in type $\widetilde{A}_2$ (outside the walls)



Theorem (Libedinsky-P. '20)

A basis of IH_y (y out of the walls, y spherical) is given by the following type of light leaves.



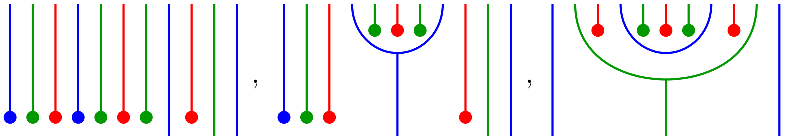
A visualization for KL polynomials

The basis that we produced give also a nice visualization of KL polynomials.

Example

We have $h_{tustusutsut,tut}(v) =$

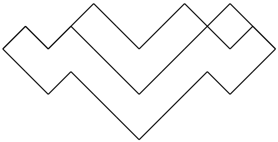
$$v^8 \quad + \quad v^6 \quad + \quad v^4$$



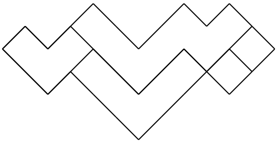
Thanks for your attention!

Dyck partitions of Type 2

A Dyck partition P is said of **type 2** if:
for any strip D that contains a box just below, SW or SE o a box in a strip C, then every box just below, SW or SE a box in C belongs either to D or C.



Type 2



Not Type 2

Remark

Between any two paths λ, μ there exists **at most one** Dyck partition of type 2.

Inverse KL polynomials

Inverse KL polynomials are related to the ordinary KL polynomials by the **inversion formula**:

$$\sum_{\mu} (-1)^{\ell(\mu)-\ell(\nu)} h_{\lambda,\mu}(v) g_{\mu,\nu}(v) = \delta_{\lambda,\nu}$$

Theorem (Brenti '02)

Dyck partitions of Type 2 describe **inverse** KL polynomials

$$\sum_{\substack{\text{P of type 2} \\ \text{between } \lambda \text{ and } \mu}} v^{|\mathsf{P}|} = g_{\lambda,\mu}(v)$$

Singular Rouquier Complexes

We can construct a complex of singular Soergel bimodules E_μ :

$$0 \to \bigoplus_{\lambda \in \Lambda_i} IH_\lambda(-i) \to \ldots \to \bigoplus_{\lambda \in \Lambda_1} IH_\lambda(-1) \to IH_\mu \to 0$$

which is **exact** everywhere but in the term IH_μ.

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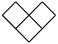
which is **exact** everywhere but in the term IH_μ.

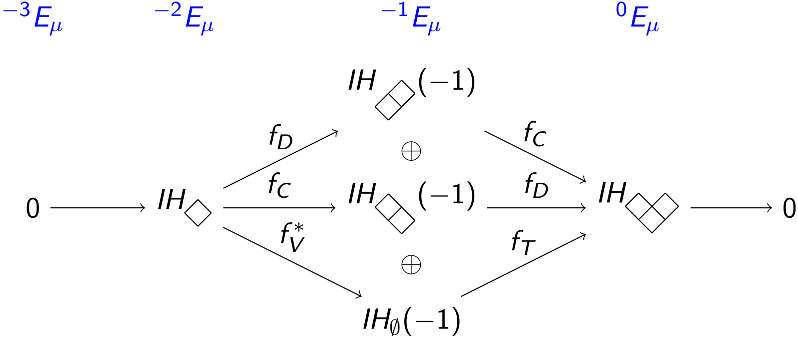
Here $\Lambda_i = \left\{ \mu \leq \lambda \mid \begin{array}{c} \text{exists Dyck partition P} \\ \text{of type 2 between } \lambda \text{ and } \mu \text{ with } |\mathrm{P}| = i \end{array} \right\}$.

$$IH_\lambda(-i) \text{ occurs in } E_\mu \quad \Longleftrightarrow \quad g_{\lambda,\mu}(v) = v^i$$

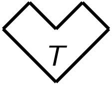
E_μ is called **singular Rouquier complex**.

Example of a singular Rouquier complex

Let $\mu =$ . The singular Rouquier complex E_μ is:



where 

and 

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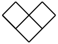
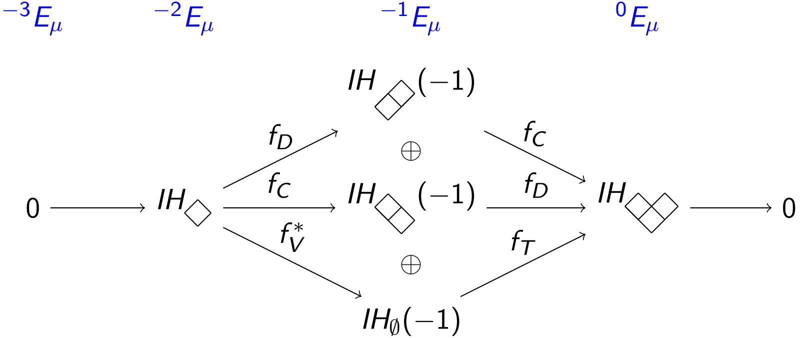
Let $\mu =$ . The singular Rouquier complex E_μ is:



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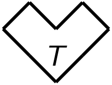
and 

.

From $d^2 = 0$ follows $f_D \circ f_C - f_C \circ f_D = f_T \circ f_V^*$ (up to scalar).